On the amplitude equations for weakly nonlinear surface waves

Sylvie Benzoni-Gavage, Jean-François Coulombel

On the amplitude equations for weakly nonlinear surface waves

Sylvie Benzoni-Gavage[†] & Jean-François Coulombel[‡]

† Université de Lyon, Université Lyon 1, INSA de Lyon, Ecole Centrale de Lyon
CNRS, UMR5208, Institut Camille Jordan, 43 blvd du 11 novembre 1918
F-69622 Villeurbanne-Cedex, France

‡ CNRS, Université Lille 1, Laboratoire Paul Painlevé (CNRS UMR8524)
and EPI SIMPAF of INRIA Lille Nord Europe
Cité scientifique, Bâtiment M2, 59655 Villeneuve d'Ascq Cedex, France

Emails: benzoni@math.univ-lyon1.fr, jfcoulom@math.univ-lille1.fr

July 8, 2011

Abstract

Nonlocal generalizations of Burgers' equation were derived in earlier work by Hunter [Contemp. Math. 1989], and more recently by Benzoni-Gavage and Rosini [Comput. Math. Appl. 2009], as weakly nonlinear amplitude equations for hyperbolic boundary value problems admitting linear surface waves. The local-in-time well-posedness of such equations in Sobolev spaces was proved by Benzoni-Gavage [Diff. Int. Eq. 2009] under an appropriate stability condition originally pointed out by Hunter. The latter stability condition has also been shown to be necessary for well-posedness in Sobolev spaces in a previous work of the authors in collaboration with Tzvetkov [Adv. Math. 2011]. In this article, we show how the verification of Hunter's stability condition follows from natural stability assumptions on the original hyperbolic boundary value problem, thus avoiding lengthy computations in each particular situation. When the original boundary value problem has a variational origin, we also show that the resulting amplitude equation has a Hamiltonian structure. Our analysis encompasses previous equations derived for nonlinear Rayleigh waves in elasticity.

AMS subject classification: 35L53, 35L50, 74B20, 35L20.
Keywords: surface waves, amplitude equation, hyperbolic equations, weakly nonlinear expansions, Hamiltonian structure, well-posedness.

Contents

1 Introduction

This article is devoted to nonlinear evolutionary boundary value problems of hyperbolic type whose linearized version admits particular solutions known as surface waves. These waves propagate along the boundary of the spatial domain and decay exponentially fast in the normal direction to the boundary. They appear in various physical contexts, such as elastodynamics - in which case they are named Rayleigh waves - or in incompressible magnetohydrodynamics. Linear surface waves approximately describe the dynamics of small amplitude solutions on a fixed time scale. One goal is to understand how the genuine nonlinear dynamics modifies the evolution of small amplitude solutions on a large time scale. In the so-called weakly nonlinear regime, this problem was addressed in a series of work originally devoted to nonlinear elastodynamics, see e.g. [20, 19, 12, 1, 6] and references therein. It has been shown in various settings that the weakly nonlinear dynamics is governed by an amplitude equation that takes the form of a nonlocal generalization of the Burgers equation. Let us observe that when the original equations are invariant by dilations, as is the case for hyperbolic systems of conservation laws, the weakly nonlinear asymptotics we are concerned with equivalently corresponds to the regime of weakly nonlinear geometric optics - which consists in studying small amplitude highly oscillating solutions on a fixed time scale.

Several questions naturally arise after the formal derivation of an amplitude equation. The first main issue is its well-posedness. It was first addressed by Hunter [12], who identified a condition for linearized stability that was later shown to be sufficient for nonlinear well-posedness [4], as well as necessary in a certain sense [9]. In the meantime, well-posedness of nonlinear hyperbolic initial boundary value problems has been well understood. So the next issue is the rigorous justification of the weakly nonlinear asymptotic expansion. An almost definitive answer has been given by Marcou in [15], the remaining gap being that she needs assume both the original nonlinear problem and the asymptotic amplitude equation to be well-posed. In other words, Hunter's stability condition is an assumption in her work. However, as more or less conjectured in [12, page 193], it seems a reasonable expectation[1] that well-posedness of the original nonlinear problem should imply well-posedness of the underlying amplitude equation. We fill in this technical gap in the present paper, and elucidate why and how well-posedness of the original nonlinear problem implies well-posedness of the amplitude equation. Our result is fairly general and gives a clear way of bypassing the technical verification of Hunter's stability condition for each particular situation, which may be a real computational challenge. The other main question that we address has to do with the algebraic structure of amplitude equations for weakly nonlinear surface waves. Hamiltonian structures of amplitude equations were pointed out in [2] in relation to scale-invariance properties of various examples of surface waves, comprising the case of Rayleigh waves in elastodynamics. Besides scale-invariance, a remarkable feature of the elastodynamics equations is their variational origin. One motivation for the present work has been to shed new light on the amplitude equation for Rayleigh waves by considering most general variational boundary problems. Under very reasonable assumptions that are of course met by the elastodynamics equations, we show indeed that the amplitude equation inherits a Hamiltonian structure. In passing we show that Hunter's stability condition readily follows from the form of the amplitude equation.

The article is organized as follows. In Section 2, we present a general framework for variational initial boundary value problems. We assume that the linearized equations admit surface waves, and derive an amplitude equation for weakly nonlinear corrections. Particular attention is paid to scale invariance of the equations and its impact on the homogeneity properties of the symbols involved in the amplitude equation. We justify the evolutionarity of the amplitude equation by making the link with the vanishing of the so-called Lopatinskii determinant. This part of our analysis is rather similar to an observation recently made in [15], see also [10]. Moreover, we prove that Hunter's stability condition holds true as soon as the amplitude equation is of evolutionary type, and exhibit a Hamiltonian structure under some technical assumptions. Section 3 is devoted to initial boundary value problems for first order hyperbolic systems, for which the derivation of the amplitude equation dates back to [12]. Our main result proves that Hunter's stability condition holds true under some natural stability conditions for the original, fully nonlinear problem. Sections 2 and 3 share some similarities but are independent from each other as they might be of interest to various audiences.

2 Hamiltonian boundary value problems

2.1 General framework

In this part, we consider boundary value problems that are governed by an energy functional of the form

$$\mathscr{E}[u] := \int_\Omega E(u, \nabla u)\,\mathrm{d}x\,,$$

[1]For instance, this expectation is fulfilled when one considers weakly nonlinear geometric optics expansions with a single phase for the Cauchy problem.

where Ω is the half-space $\{x = (x_1, \ldots, x_d)\,;\ x_d > 0\}$ of $\mathbb{R}^d$, the unknown function u takes its values in $\mathbb{R}^N$, ∇u is the matrix-valued function of coefficients $\partial_j u_\alpha$, with u_α the αth component of u for $\alpha \in \{1, \ldots, N\}$ and ∂_j denoting the partial derivative with respect to x_j for $j \in \{1, \ldots, d\}$. We assume that the mapping

$$\begin{array}{rlcl} E: & \mathbb{R}^N \times \mathbb{R}^{N\times d} & \to & \mathbb{R} \\ & (u, F) & \mapsto & E(u, F) \end{array}$$

is 'smooth' enough[2]. We shall denote by $F_{\alpha j}$ the coefficients of a matrix $F \in \mathbb{R}^{N\times d}$, and use repeatedly, without further warning, Einstein's convention of summation over repeated indices. Let us introduce the notations

$$a_{\alpha\beta} := \frac{\partial^2 E}{\partial u_\alpha \partial u_\beta}\,, \quad b_{\alpha\beta j} := \frac{\partial^2 E}{\partial u_\alpha \partial F_{\beta j}}\,, \quad c_{\alpha j\beta\ell} := \frac{\partial^2 E}{\partial F_{\alpha j}\partial F_{\beta\ell}}\,. \tag{1}$$

Note that if $b_{\alpha\gamma m} \equiv 0$ for all $\alpha, \gamma \in \{1, \ldots, N\}$ and all $m \in \{1, \ldots, d\}$, then $a_{\alpha\beta}$ depends only on u and $c_{\alpha j\beta\ell}$ depends only on F. This is what happens when the energy E is in separate form $E(u, F) = E_0(u) + W(F)$, which is the case for instance in wave/elasticity equations (see (5) below).

Our primary assumption is the following.

(H0) Legendre–Hadamard condition:

$$\sum_{\alpha,\beta,j,\ell} c_{\alpha j\beta\ell}(u, F)\, v_\alpha\, v_\beta\, \xi_j\, \xi_\ell \,\geq\, 0\,, \quad \forall F \in \mathbb{R}^{N\times d}\,,\ \forall v \in \mathbb{R}^N\,,\ \forall \xi \in \mathbb{R}^d\,, \tag{2}$$

for u in the neighborhood of some $\underline{u}$, ensuring that E is (locally) *rank 1-convex* with respect to F.

We easily see, at least formally, that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\,\mathscr{E}[u + \theta h]_{|\theta=0} \,=\, \int_\Omega \left(\frac{\partial E}{\partial u_\alpha}(u, \nabla u) \,-\, D_j \left(\frac{\partial E}{\partial F_{\alpha j}}(u, \nabla u) \right) \right) h_\alpha \,\mathrm{d}x \\ +\, \int_{\partial\Omega} \frac{\partial E}{\partial F_{\alpha j}}(u, \nabla u)\, n_j\, h_\alpha \,\mathrm{d}\check{x}\,,$$

where D_j denotes the total derivative with respect to x_j, n_j is the jth component of the exterior unit normal $\mathbf{n}$ to $\partial\Omega$, which actually reduces to $\mathbf{n} = (0, \ldots, 0, -1)$, and $\check{x} := (x_1, \ldots, x_{d-1})$. In a more compact form,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\,\mathscr{E}[u + \theta h]_{|\theta=0} \,=\, \int_\Omega \delta\mathscr{E}[u] \cdot h \,\mathrm{d}x \,+\, \int_{\partial\Omega} \delta_{\mathbf{n}}\mathscr{E}[u] \cdot h \,\mathrm{d}\check{x}\,,$$

where $\delta\mathscr{E}[u]$ is the usual variational gradient of $\mathscr{E}$, of components

$$\begin{aligned} \delta\mathscr{E}[u]_\alpha &:= \frac{\partial E}{\partial u_\alpha}(u, \nabla u) \,-\, D_j \left(\frac{\partial E}{\partial F_{\alpha j}}(u, \nabla u) \right) \\ &= \frac{\partial E}{\partial u_\alpha}(u, \nabla u) \,-\, b_{\beta\alpha j}(u, \nabla u)\, \partial_j u_\beta \,-\, c_{\alpha j\beta\ell}(u, \nabla u)\, \partial_j \partial_\ell u_\beta\,, \end{aligned}$$

[2] at least $\mathscr{C}^3$

and

$$\delta_{\mathbf{n}}\mathscr{E}[u]_\alpha \,:=\, \frac{\partial E}{\partial F_{\alpha j}}(u,\nabla u)\,n_j \;=\; -\,\frac{\partial E}{\partial F_{\alpha d}}(u,\nabla u)$$

if we take into account our choice of coordinates, which is such that $\mathbf{n} = (0,\dots,0,-1)$. In particular, a sufficiently smooth critical point u of $\mathscr{E}$ must satisfy the *elliptic boundary value problem*

$$\delta\mathscr{E}[u] \,=\, 0_N \ \text{ in } \ \Omega\,, \quad \delta_{\mathbf{n}}\mathscr{E}[u] \,=\, 0_N \ \text{ on } \ \partial\Omega\,. \tag{3}$$

Here above, 0_N stands for the null vector of $\mathbb{R}^N$. When no confusion can occur we will omit the subscript N. Not that the ellipticity of the Euler–Lagrange equation $\delta\mathscr{E}[u] \,=\, 0$ comes from the Legendre–Hadamard assumption (2).

We are in fact interested in dynamical versions of the boundary value problem (3), and more precisely in *Hamiltonian boundary value problems* of the form

$$\partial_t u \,=\, J\,\delta\mathscr{E}[u] \ \text{ in } \ \Omega\,, \quad \delta_{\mathbf{n}}\mathscr{E}[u] \,=\, 0 \ \text{ on } \ \partial\Omega\,, \tag{4}$$

where $J \in \mathbb{R}^{N\times N}$ is a skew-symmetric (constant) matrix[3]. In addition, we assume that J is non-singular (which together with skew-symmetry obviously requires that N be even, but this will play no role in what follows). Dynamical systems of the form $\partial_t u = J\,\delta\mathscr{E}[u]$ arise for instance as the Euler–Lagrange equations associated with space-time functionals

$$\mathscr{G}[\chi] \,:=\, \int_{\mathbb{R}}\int_\Omega \tfrac{1}{2}\,|\partial_t\chi|^2 \,-\, W(\nabla_x\chi)\,\mathrm{d}x\,\mathrm{d}t\,.$$

Indeed, we have

$$\delta\mathscr{G}[\chi]_\alpha \,=\, -\,\partial_t^2\chi_\alpha \,+\, D_j\left(\frac{\partial W}{\partial F_{\alpha j}}(\nabla_x\chi)\right)\,,$$

so that, by setting $p := \partial_t\chi$, we see that the Euler–Lagrange equation $\delta\mathscr{G}[\chi] \,=\, 0$ is equivalent to

$$\partial_t\left(\begin{array}{c}\chi\\ p\end{array}\right) \,=\, \left(\begin{array}{cc}\mathbf{0}_n & I_n\\ -I_n & \mathbf{0}_n\end{array}\right)\delta\mathscr{E}[\chi,p]\,,\quad \mathscr{E}[\chi,p] \,:=\, \int_\Omega \tfrac{1}{2}|p|^2 \,+\, W(\nabla_x\chi)\,\mathrm{d}x\,, \tag{5}$$

where n is the number of components of the vector-valued unknown χ, and $\mathbf{0}_n$ denotes the null $n\times n$ matrix while I_n is the $n\times n$ identity matrix. Prototypes of such equations are

- for $n = 1$, wave equations,
- for $n = d$, elasticity equations.

In particular, when W is quadratic the Euler–Lagrange equation $\delta\mathscr{G}[\chi] = 0$ is linear. The boundary value problem for this kind of equation with the - also linear - homogeneous boundary conditions

$$\frac{\partial W}{\partial F_{\alpha j}}(\nabla\chi)\,n_j \,=\, 0$$

[3]Of course, more general Hamiltonian systems, with J a differential operator, are also of interest, for instance if we are to consider incompressible Euler equations, or incompressible MHD equations, but they can hardly be studied as a whole.

(which are nothing but $\delta_{\mathbf{n}}\mathscr{E}[u] = 0$ with our notations) has been investigated in detail by Serre [23]: under a strict rank-1 convexity assumption, he proved the equivalence between the well-posedness of the boundary value problem and the existence of families of *linear surface waves*, that is, special solutions of the form

$$\chi(t,x) = \mathrm{e}^{i(\eta_0 t + \check{\eta}\cdot\check{x})}\, X(x_d)\,,$$

where $\eta_0 \in \mathbb{R}$ (a temporal frequency), $\check{\eta} = (\eta_1, \dots, \eta_{d-1}) \in \mathbb{R}^{d-1}$ (a wave vector in the boundary direction), and $X \to 0$ exponentially fast when $x_d \to +\infty$, see Theorem 1 below for more details.

Our purpose here is to address nonlinear Hamiltonian boundary value problems (4), with most general energy functionals $\mathscr{E}$. Still, we shall restrict to problems whose linearization about a point $(\underline{u}, 0)$ is *scale invariant*. The motivation for this restriction, which we shall make more precise below (see Section 2.2.1), is to enforce not only one linear surface wave propagating along the boundary as $\mathrm{e}^{i(\eta_0 t + \check{\eta}\cdot\check{x})}$, but a cone of linear surface waves, associated with neutral modes of the form $\mathrm{e}^{ik(\eta_0 t + \check{\eta}\cdot\check{x})}$ for all nonzero real number k. In this situation, we may look for *weakly nonlinear surface waves* as higher order approximations of the Hamiltonian boundary value problem in (4). Consistently with earlier work by Parker *et al* [19, 20] regarding elasticity, Hunter [12] for first order systems with linear boundary conditions, Alì and Hunter [1] for MHD, and Benzoni-Gavage *et al* [6] for phase transitions, we are going to show that those weakly nonlinear surface waves are governed by *amplitude equations* that are nonlocal generalizations of the Burgers equation. To be more precise, we shall justify the evolutionarity of these equations by showing that the coefficient of the time derivative is nonzero under a 'generic' assumption[4]. This confirms, in a different framework, an observation recently made by Marcou [15] on dissipative, first order boundary value problems. Our other contribution in this part is twofold. We show that the amplitude equations derived from Hamiltonian boundary value problems always satisfy Hunter's stability condition [12]. This somehow natural condition has been shown to be necessary and sufficient for the well-posedness of the Cauchy problem for nonlocal Burgers equations [13, 15, 4, 9]. However, it looks very tricky to check in general. We shall come back to it, and elucidate its verification, in Section 3.4 for first order problems. For Hamiltonian problems, it turns out that Hunter's stability condition can be shown in a rather straightforward manner. Furthermore, we show that those amplitude equations inherit a *Hamiltonian structure*. This is consistent with a classical 'rule of thumb' formulated and justified on water wave equations by Olver [17]. Similar Hamiltonian structures were pointed out by dimensional analysis in [2]. However, up to our knowledge, their derivation from a general, fully nonlinear problem is new.

2.2 Main notations and assumptions

2.2.1 Scale invariance

Recall that Ω is a half-space, so that the homogeneous boundary condition $\delta_{\mathbf{n}}\mathscr{E}[u] = 0$ on $\partial\Omega$ has a chance to be invariant by spatial dilations.

[4]Namely, that the so-called Lopatinskii determinant vanishes exactly at first order.

Before investigating the scale invariance of the linearized boundary value problem, let us introduce some more notations. First of all, we take a look at the second order variation of $\mathscr{E}$. We have

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\,\mathscr{E}[u+\theta h]_{|\theta=0} \;=\; \int_\Omega (\mathscr{L}[u]\,h)\cdot h\,\mathrm{d}x \;+\; \int_{\partial\Omega} (\mathscr{C}_{\mathbf{n}}[u]\,h)\cdot h\,\mathrm{d}\check{x}\,,$$

where $\mathscr{L}[u] := \mathrm{Hess}\mathscr{E}[u]$ is the vector-valued second order differential operator defined by

$$(\mathscr{L}[u]\,h)\cdot h' \;=\; (a_{\alpha\beta}\,h_\beta \;+\; b_{\alpha\beta j}\,\partial_j h_\beta \;-\; D_j(b_{\beta\alpha j}\,h_\beta \;+\; c_{\alpha j\beta\ell}\,\partial_\ell h_\beta)\,)\,h'_\alpha\,, \tag{6}$$

and $\mathscr{C}_{\mathbf{n}}[u]$ is the vector-valued first order differential operator defined by

$$(\mathscr{C}_{\mathbf{n}}[u]\,h)\cdot h' \;=\; (b_{\beta\alpha j}\,h_\beta \;+\; c_{\alpha j\beta\ell}\,\partial_\ell h_\beta)\,n_j\,h'_\alpha \;=\; -\,(b_{\beta\alpha d}\,h_\beta \;+\; c_{\alpha d\beta\ell}\,\partial_\ell h_\beta)\,h'_\alpha \tag{7}$$

when $\mathbf{n} = (0,\ldots,0,-1)$. In the sequel we omit the subscript $\mathbf{n}$. Here above, all terms $a_{\alpha\beta}$, $b_{\alpha\beta j}$, and $c_{\alpha j\beta\ell}$ are evaluated at $(u,\nabla u)$. Recall also that D_j denotes the total derivative with respect to x_j. The dot between two vectors of $\mathbb{R}^N$ means the usual inner product. The operators $\mathscr{L}[u]$ and $\mathscr{C}[u]$ are the building blocks of linearized boundary value problems. More precisely, if a constant state $\underline{u}\in\mathbb{R}^N$ is such that $(\underline{u},0)$ is a critical point of E, then $u\equiv\underline{u}$ is a special solution of (4), and linearizing (4) about $(\underline{u},0)$ we get the linear boundary value problem

$$\partial_t u \;=\; J\,\mathscr{L}[\underline{u}]\,u \;\text{ for }\; x_d>0\,, \quad \mathscr{C}[\underline{u}]\,u \;=\; 0\,, \;\text{ at } x_d=0\,. \tag{8}$$

By substituting $(\underline{u},0)$ for $(u,\nabla u)$ in (6)-(7), we see that

$$(\mathscr{L}[\underline{u}]\,u)_\alpha \;=\; \underline{a}_{\alpha\beta}\,u_\beta \;+\; (\underline{b}_{\alpha\beta j} \;-\; \underline{b}_{\beta\alpha j})\,\partial_j u_\beta \;-\; \underline{c}_{\alpha j\beta\ell}\,\partial_j\partial_\ell u_\beta\,,$$

$$(\mathscr{C}[\underline{u}]\,h)_\alpha \;=\; -\,(\underline{b}_{\beta\alpha d}\,u_\beta \;+\; \underline{c}_{\alpha d\beta\ell}\,\partial_\ell u_\beta)\,,$$

where underlined letters $\underline{a}$, $\underline{b}$, $\underline{c}$ stand for a, b, c evaluated at $(\underline{u},0)$. Whether (8) is scale invariant is not obvious. Clearly, it is not invariant merely by space-time dilations $(x,t)\mapsto(kx,kt)$, $k>0$, in general. By contrast with first-order systems considered in Section 3, a change of scale of the unknown u is needed here. This is easily seen for example on (the linearized version of) (5), in which $p=\partial_t\chi$. Indeed, if we set $\widetilde{x}=kx$, $\widetilde{t}=kt$, since the displacement χ has the dimension of a length and thus scales as x it is natural to set on the one hand $\widetilde{\chi}(\widetilde{x},\widetilde{t}\,)=k\chi(x,t)$. This ensures that $\nabla_{\widetilde{x}}\widetilde{\chi} = \nabla_x\chi$ and $\partial_{\widetilde{t}}\widetilde{\chi}=\partial_t\chi$, hence the invariance of the (local) potential energy $W(\nabla_x\chi)$ in general and *a fortiori* when W is quadratic. On the other hand, no change of scale is needed for p because it has the dimension of a velocity, and velocities are obviously unchanged by the scaling considered here. By setting $\widetilde{p}(\widetilde{x},\widetilde{t}\,)=p(x,t)$ we readily infer from $p=\partial_t\chi$ the identical relationship $\widetilde{p}=\partial_{\widetilde{t}}\widetilde{\chi}$. We thus see that the scaled system, in the variables $(\widetilde{x},\widetilde{t},\widetilde{\chi},\widetilde{p})$, is identical (when we remove the tildes) to the original one in the variables (x,t,χ,p). This observation is even valid for the nonlinear system in (5).

We are now going to concentrate on the scale invariance of the linearized version (8) of the abstract system (4). As conventions vary from one community to the other, let us stress that by scale invariance we mean here that new independent variables

$(\widetilde{x}, \widetilde{t}) := (kx, kt)$ together with new dependent variables of the form $\widetilde{u}_\alpha := k^{\theta_\alpha} u_\alpha$, viewed as functions of $(\widetilde{x}, \widetilde{t})$, leave the system invariant[5] for a suitable choice of the rational numbers θ_α's independent of $k \neq 0$. In the case of (8), scale invariance is characterized as follows.

Lemma 1. *Let us a consider a change of scale of the form*

$$(x, t, u_1, \ldots, u_N) \mapsto (kx, kt, k^{\theta_1} u_1, \ldots, k^{\theta_N} u_N)\,,$$

which can be viewed as a symmetry of infinitesimal generator

$$\mathbf{v} := x_i \partial_{x_i} \,+\, t\,\partial_t \,+\, \theta_\alpha\, u_\alpha\, \partial_{u_\alpha}\,. \tag{9}$$

The boundary value problem (8) *is invariant with respect to the symmetry group generated by* $\mathbf{v}$ *if and only if, for all* $\alpha, \beta \in \{1, \ldots, N\}$,

$$(-1 \,+\, \theta_\alpha \,-\, \theta_\beta)\, J_{\alpha\gamma}\, \underline{a}_{\gamma\beta} \,=\, 0\,, \tag{10}$$

$$(\theta_\alpha - \theta_\beta)\, J_{\alpha\gamma}\, (\underline{b}_{\beta\gamma j} \,-\, \underline{b}_{\gamma\beta j}) \,=\, 0 \quad \forall j \in \{1, \ldots, d\}\,, \tag{11}$$

$$(1 \,+\, \theta_\alpha \,-\, \theta_\beta)\, J_{\alpha\gamma}\, \underline{c}_{\gamma j \beta \ell} \,=\, 0 \quad \forall j, \ell \in \{1, \ldots, d\} \tag{12}$$

$$(\theta_\beta - \theta^\alpha)\, \underline{b}_{\beta\alpha d} \,=\, 0 \;\; \text{and}\; (\theta_\beta - \theta^\alpha - 1)\, \underline{c}_{\beta\alpha d\ell} = 0\;,\;\; \forall \ell \in \{1, \ldots, d\}\,, \tag{13}$$

for some rational numbers θ^α, $\alpha \in \{1, \ldots, N\}$.

(In Eqs (10)-(13) here above, the summation convention is not to be applied to the indices α, β.) The proof is postponed to the appendix, see Lemma A.1. It is to be noted that, as far as we are concerned with the scale invariance of a *linear* system, we can add an arbitrary number θ to all the θ_α's (which amounts to multiplying u by k^θ). Since the exponents θ_α's only occur in differences, the above conditions (10)-(11)-(12)-(13) are of course preserved by the addition of any θ to all the θ_α's, up to also adding θ to the θ^α's in (13).

We can easily check that the above conditions (10)-(11)-(12)-(13) are satisfied with $\theta_\alpha = 1$ for $\alpha \leq n$ and $\theta_\alpha = 0$ for $\alpha \geq n+1$ for the linearized version of (5), in which $N = 2n$,

$$J \,=\, \begin{pmatrix} \mathbf{0}_n & I_n \\ -I_n & \mathbf{0}_n \end{pmatrix}\,,\quad \underline{a}_{\gamma\beta} \,=\, 0 \text{ if } \gamma \text{ or } \beta \leq n\,,\quad \underline{c}_{\gamma j \beta k} \,=\, 0 \text{ if } \gamma \text{ or } \beta \geq n+1\,,$$

and $\underline{b}_{\beta\gamma j} \,=\, 0$ for all β, γ, j. This is consistent with the observation made above that the natural rescaling is $(x, t, \chi, p) \mapsto (kx, kt, k\chi, p)$.

Another example of scale-invariant Hamiltonian system is given by the compressible 3D Euler equations written in Clebsch coordinates. This is a not so well-known formulation of the Euler equations, which is based on the Clebsch decomposition of the velocity field $\mathbf{u}$ of the fluid as $\mathbf{u} \,=\, \nabla\varphi \,+\, \lambda\,\nabla\mu$ (see [3, § 7] for more details). If ρ denotes the density

[5] In other words, removing the tildes from both the independent variables and the dependent variables in the scaled system gives back the original one.

of the fluid, the unknown functions are then chosen to be $(\rho, \Lambda := \rho\lambda, \varphi, \mu)$ instead of (ρ, u_1, u_2, u_3), and such a way that the total energy is of the form

$$E(\rho, \Lambda, \nabla\varphi, \nabla\mu) \;=\; P(\rho) \,+\, \frac{1}{2}\,\rho\,\|\nabla\varphi + \tfrac{\Lambda}{\rho}\,\nabla\mu\|^2\,. \tag{14}$$

We have assumed for simplicity that the fluid is either isothermal or adiabatic, so that P denotes either the free energy or the internal energy per unit volume. In both cases the pressure of the fluid is

$$p = p(\rho) = \rho P'(\rho) - P(\rho)\,.$$

Setting $\mathscr{E}[U] := \int_\Omega E(\rho, \Lambda, \nabla\varphi, \nabla\mu)\,\mathrm{d}x$ with $U := (\rho, \Lambda, \varphi, \mu)^T$, it is not difficult to check that the conservation laws of mass and momentum are (formally) equivalent to the Hamiltonian system

$$\partial_t U \;=\; J\,\delta\mathscr{E}[U]\,, \quad J \;=\; \begin{pmatrix} \mathbf{0}_2 & I_2 \\ -I_2 & \mathbf{0}_2 \end{pmatrix}\,. \tag{15}$$

This system is unsurprisingly scale-invariant, since the Euler equations are so. More precisely, the Euler equations are invariant by the rescaling

$$(x, t, \rho, u_1, u_2, u_3) \mapsto (kx, kt, \rho, u_1, u_2, u_3)\,,$$

so we expect the above system to be invariant by $(x, t, \rho, \Lambda, \varphi, \mu) \mapsto (kx, kt, \rho, \Lambda, k\varphi, k\mu)$. This is indeed the case that (10)-(11)-(12)-(13) are satisfied with $\theta_1 = 0$, $\theta_2 = 0$, $\theta_3 = 1$, $\theta_4 = 1$ (which imply $1 + \theta_{1,2} - \theta_{3,4} = 0$) because the form of the energy implies that $\underline{a}_{\gamma\beta} = 0$ for all α, β except for $\underline{a}_{11} = P''(\underline{\rho})$, $\underline{b}_{\beta\gamma j} = 0$ for all β, γ, j, and $\underline{c}_{\gamma j\beta k} = 0$ if γ or $\beta \leq 2$.

Let us go back to our abstract system (8). From now on, we make the following assumption.

(H1) Scale invariance: There exist rational numbers θ_α such that the linearized boundary value problem (8) is invariant by the rescaling

$$(x, t, u_1, \ldots, u_N) \mapsto (kx, kt, k^{\theta_1}u_1, \ldots, k^{\theta_N}u_N)\,,$$

for all $k > 0$. Furthermore, we assume for simplicity that[6] $\underline{b}_{\beta\gamma j} = 0$ for all β, γ, j. Scale-invariance thus reduces to (10)(12) together with the second part in (13), for all α and $\beta \in \{1, \ldots, N\}$.

2.2.2 Linear surface waves

A necessary condition for (8) to be well-posed in the usual sense for hyperbolic boundary value problems is that it does not admit any exponentially growing mode of the form

$$u(t, x) = \mathrm{e}^{\tau t + i\,\check{\eta}\cdot\check{x} - \omega\,x_d}\,u_0\,, \quad \mathsf{Re}\,\tau > 0\,,\; \check{\eta} \in \mathbb{R}^{d-1}\,,\; \mathsf{Re}\,\omega > 0\,,\, u_0 \in \mathbb{C}^N\backslash\{0\}\,.$$

This can be phrased in terms of the ODE problem obtained from (8) by Fourier–Laplace transform in $(t, \check{x})$,

$$\tau\,\widehat{u} \;=\; J\,\mathscr{L}_{\underline{u}}^{\check{\eta}}\,\widehat{u} \;\text{ for }\; z > 0\,, \quad \mathscr{C}^{\check{\eta}}\,\widehat{u} \;=\; 0 \;\text{ at } z = 0\,, \tag{16}$$

[6] as is the case in the examples above

where $z = x_d$ and

$$(\mathscr{L}^{\check{\eta}}_{\underline{u}} h)_\alpha \;=\; (\underline{a}_{\alpha\beta} + \underline{c}_{\alpha j\beta\ell}\,\eta_j\,\eta_\ell)\,h_\beta - i\,(\underline{c}_{\alpha j\beta d}\,\eta_j + \underline{c}_{\alpha d\beta\ell}\,\eta_\ell)\,\partial_z h_\beta - \underline{c}_{\alpha d\beta d}\,\partial_z^2 h_\beta\,, \tag{17}$$

$$(\mathscr{C}^{\check{\eta}}\,h)_\alpha \;=\; -\,i\,\underline{c}_{\alpha d\beta\ell}\,\eta_\ell\,h_\beta - \underline{c}_{\alpha d\beta d}\,\partial_z h_\beta\,. \tag{18}$$

(H2) Lopatinskii stability condition: For all τ with $\mathsf{Re}\,\tau > 0$, for all $\check{\eta} \in \mathbb{R}^{d-1}$, the boundary value problem in (16) has no nontrivial, square integrable solution.

This assumption will not be used explicitly but if it were not satisfied then the boundary value problem would be ill-posed, and what follows about the weakly nonlinear regime would be irrelevant. Here we are going to concentrate on what happens for purely imaginary values of τ. Before going further, let us introduce the $N\times N$ matrices

$$\Gamma_{\underline{u}} := (\underline{a}_{\alpha\beta})\,,\quad \Lambda := (\underline{c}_{\alpha d\beta d})\,,\quad A^{\check{\eta}} := (\eta_\ell\,\underline{c}_{\alpha d\beta\ell})\,,\quad \Sigma^{\check{\eta}} := (\eta_j\eta_\ell\,\underline{c}_{\alpha j\beta\ell})\,, \tag{19}$$

in a such a way that

$$\mathscr{L}^{\check{\eta}}_{\underline{u}} \;=\; \Gamma_{\underline{u}} + \Sigma^{\check{\eta}} - i\,(A^{\check{\eta}} + (A^{\check{\eta}})^T)\,\partial_z - \Lambda\,\partial_z^2\,, \tag{20}$$

$$\mathscr{C}^{\check{\eta}} \;=\; -\,i\,A^{\check{\eta}} - \Lambda\,\partial_z\,. \tag{21}$$

With these notations, (16) equivalently reads

$$\left\{\begin{array}{ll} \tau\,\widehat{u} \;=\; J\,(\Gamma_{\underline{u}} + \Sigma^{\check{\eta}})\,\widehat{u} - i\,J\,(A^{\check{\eta}} + (A^{\check{\eta}})^T)\,\partial_z\widehat{u} - J\,\Lambda\,\partial_z^2\widehat{u} & \text{for } \; z>0\,,\\ i\,A^{\check{\eta}}\widehat{u} + \Lambda\,\partial_z\widehat{u} \;=\; 0 & \text{at } z=0\,. \end{array}\right. \tag{22}$$

In what follows, we will be considering (22) with $\tau = i\eta_0 \in i\mathbb{R}$. In particular we are going to discuss whether this ODE boundary value problem inherits a Hamiltonian structure. With the matrix notations introduced in (19), the Legendre–Hadamard condition in (2) at $(u,F) = (\underline{u},0)$ equivalently reads

$$\xi^2\,\Lambda + \xi\,(A^{\check{\eta}} + (A^{\check{\eta}})^T) + \Sigma^{\check{\eta}} \;\geq\; 0\,,\quad \forall \xi\in\mathbb{R}\,,\ \forall\check{\eta}\in\mathbb{R}^{d-1}\,. \tag{23}$$

In particular, it implies that the matrix Λ is real symmetric and non-negative.

Assume for a while that Λ is non-singular. Then it is not difficult to see that (22) with $\tau = i\eta_0$ admits the nice Hamiltonian formulation

$$\partial_z \begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix} = \mathbf{J}\,\mathbf{H}^{\eta}_{\underline{u}} \begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix} \;\text{ for }\; z>0\,,\quad \widehat{v}_{|z=0} \;=\; 0\,, \tag{24}$$

where

$$\mathbf{J} \;:=\; \begin{pmatrix} \mathbf{0}_N & I_N \\ -I_N & \mathbf{0}_N \end{pmatrix}$$

is obviously skew-symmetric, and

$$\mathbf{H}^{\eta}_{\underline{u}} \;:=\; \left(\begin{array}{c|c} -\Gamma_{\underline{u}} - \Sigma^{\check{\eta}} + i\eta_0 J^{-1} + (A^{\check{\eta}})^T\Lambda^{-1}A^{\check{\eta}} & i(A^{\check{\eta}})^T\Lambda^{-1} \\ \hline -i\Lambda^{-1}A^{\check{\eta}} & \Lambda^{-1} \end{array}\right)$$

is Hermitian since $\Gamma_{\underline{u}}$, $\Sigma^{\check\eta}$, and Λ are real symmetric, and $i\eta_0 J^{-1}$ is Hermitian.

However, the invertibility of Λ is false in both examples discussed in Section 2.2.1 above. For the first one, that is for an energy $\mathscr{E}$ as in (5), we have the following block structures

$$J = \begin{pmatrix} \mathbf{0}_n & I_n \\ -I_n & \mathbf{0}_n \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \breve\Lambda & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n \end{pmatrix}, \quad \Gamma_{\underline{u}} = \begin{pmatrix} \mathbf{0}_n & \mathbf{0}_n \\ \mathbf{0}_n & I_n \end{pmatrix},$$

$$A^{\check\eta} = \begin{pmatrix} \breve A^{\check\eta} & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n \end{pmatrix}, \quad \Sigma^{\check\eta} = \begin{pmatrix} \breve\Sigma^{\check\eta} & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n \end{pmatrix}.$$

In particular, Λ is obviously singular. Nevertheless, if we assume that $\breve\Lambda$ is nonsingular (which means that the rank of Λ is exactly equal to $n = N/2$, and is indeed the case under the strict rank-1 convexity condition that will be stated in (34) below), the boundary value problem in (22) still admits a nice formulation analogous to (24). Indeed, the interior equation in (22) equivalently reads

$$\widehat{p} = \tau\,\widehat{\chi} \;\text{ and }\; -\tau^2\,\widehat{\chi} = (\breve\Sigma^{\check\eta} - i\,(\breve A^{\check\eta} + (\breve A^{\check\eta})^T)\,\partial_z - \breve\Lambda\,\partial_z^2)\widehat{\chi}\,,$$

while the boundary condition at $z = 0$ reads

$$(i\,\breve A^{\check\eta} + \breve\Lambda\,\partial_z)\,\widehat{\chi} = 0\,.$$

Therefore, the boundary value problem in (22) with $\tau = i\eta_0$ is equivalent to

$$\partial_z \begin{pmatrix} \widehat{\chi} \\ \widehat{\psi} \end{pmatrix} = J\,H^\eta \begin{pmatrix} \widehat{\chi} \\ \widehat{\psi} \end{pmatrix} \;\text{ for }\; z > 0\,, \quad \widehat{\psi}_{|z=0} = 0\,, \tag{25}$$

where J is the same skew-symmetric matrix as before, and

$$H^\eta := \left(\begin{array}{c|c} -\breve\Sigma^{\check\eta} + \eta_0^2 I_n + (\breve A^{\check\eta})^T \breve\Lambda^{-1} \breve A^{\check\eta} & i(\breve A^{\check\eta})^T \breve\Lambda^{-1} \\ \hline -i\breve\Lambda^{-1}\breve A^{\check\eta} & \breve\Lambda^{-1} \end{array}\right) \tag{26}$$

is Hermitian since $\breve\Sigma^{\check\eta}$ and $\breve\Lambda$ are real symmetric. As regards the second example, namely the Euler equations in Clebsch coordinates (15), with E as in (14), the situation is more degenerate. Indeed, the linearized problem in (8) amounts in this case to the *acoustic* equations with the characteristic boundary condition at $z = 0$ that the normal velocity be zero. Under the conditions $\underline{\varrho} > 0$, $P''(\underline{\varrho}) > 0$, we find that (22) with $\tau = i\eta_0$ is equivalent to $\widehat{\rho} = -i\eta_0\,\widehat{\varphi}/P''(\underline{\varrho})$, $\widehat{\Lambda} = \underline{\Lambda}\,\widehat{\rho}/\underline{\varrho}$ (this Λ being one of the unknowns has nothing to do with the matrix of the same name), $\widehat{\mu} = 0$ (this simple relation being reminiscent of the fact that the time derivative of the velocity field is potential in solutions of the acoustic equations), and

$$\partial_z \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \underline{c}^2\eta_0^2 - \|\check\eta\|^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} \;\text{ for }\; z > 0\,, \quad \widehat{\psi}_{|z=0} = 0\,, \tag{27}$$

where $\underline{c} := \sqrt{\underline{\rho}\, P''(\underline{\rho})} = \sqrt{p'(\underline{\rho})}$ denotes the sound speed. We thus see that, despite the degeneracy of the problem, there remains something of the original PDEs Hamiltonian structure in the linearized boundary value problem. This is in accordance with a 'rule of thumb' saying that Hamiltonian structures are robust enough to persist under many kinds of transformations. To support this idea, let us show a rather general result that comprises at least all three cases considered above, namely, the abstract case with an invertible Λ, the case (5) containing wave/elasticity equations, and compressible fluid equations in Clebsch coordinates.

Proposition 1. *Let η_0 be a real number, and J, S, A, Λ be real $N\times N$ matrices. Assuming that J is nonsingular and skew-symmetric, and that S and Λ are symmetric, we consider the ODE problem*

$$\left\{\begin{array}{ll} i\eta_0\, u \,=\, J\,S\,u \,-\, i\,J\,(A+A^T)\,\partial_z u \,-\, J\,\Lambda\,\partial_z^2 u & \textit{for } \ z>0\,,\\ i\,Au \,+\, \Lambda\,\partial_z u \,=\, 0 & \textit{at } z=0\,, \end{array}\right. \tag{28}$$

where the unknown u takes its values in $\mathbb{C}^N$. Let p be the rank of Λ. If $p<N$ we assume in addition that

$$\ker\Lambda \subset \ker A\,, \quad \ker\Lambda \subset \ker A^T\,, \tag{29}$$

and, denoting by Π_0 the orthogonal projector onto the kernel of Λ, that

$$\Pi_0 (S\,-\,i\eta_0 J^{-1})\Pi_0 \ \ \textit{is an isomorphism of } \ker\Lambda\,. \tag{30}$$

Then there exists a unitary matrix $P\in\mathbb{C}^{N\times N}$, a Hermitian matrix $\mathbf{H}\in\mathbb{C}^{(2p)\times(2p)}$, and a rectangular matrix $Q\in\mathbb{C}^{(N-p)\times p}$ such that (28) is equivalent to

$$u \,=\, P\begin{pmatrix}\breve{u}\\ Q\breve{u}\end{pmatrix} \quad \textit{for } \ z\geq 0\,, \tag{31}$$

$$\partial_z\begin{pmatrix}\breve{u}\\ \breve{v}\end{pmatrix} \,=\, \mathbf{J}\,\mathbf{H}\begin{pmatrix}\breve{u}\\ \breve{v}\end{pmatrix} \quad \textit{for } \ z>0\,, \tag{32}$$

$$\breve{v} \,=\, 0 \quad \textit{at } z=0\,, \tag{33}$$

with

$$\mathbf{J} \,:=\, \begin{pmatrix}\mathbf{0}_p & I_p\\ -I_p & \mathbf{0}_p\end{pmatrix}\,.$$

Proof. We have already dealt with the case $p=N$ (in which there is no matrix Q, and $P=I_N$). Assume that p is less than N. Our assumptions in (29) ensure that

$$A\Pi_0 = 0\,,\ A^T\Pi_0 = 0\,,$$

hence also

$$\Pi_0 A^T = 0\,,\ \Pi_0 A = 0\,,$$

by taking the adjoints (the orthogonal projector Π_0 being self-adjoint). Therefore, (28) is equivalent to

$$\left\{\begin{array}{ll} v = i\,A\Pi_1 u \,+\, \Lambda\Pi_1\,\partial_z u & \text{for } \ z>0\,,\\ i\eta_0\,\Pi_1 J^{-1}\,(\Pi_0 u \,+\, \Pi_1 u) \,=\, \Pi_1\,S\,(\Pi_0 u \,+\, \Pi_1 u) \,-\, i\,\Pi_1\,A^T\,\Pi_1\partial_z u \,-\, \Pi_1\partial_z v & \text{for } \ z>0\,,\\ i\eta_0\,\Pi_0\,J^{-1}\,(\Pi_0 u \,+\, \Pi_1 u) \,=\, \Pi_0\,S\,(\Pi_0 u \,+\, \Pi_1 u) & \text{for } \ z>0\,,\\ v \,=\, 0 & \text{at } z=0\,, \end{array}\right.$$

where $\Pi_1 := I_N - \Pi_0$. Equivalently, using that $\Pi_1\Pi_1 = \Pi_1$, $\Pi_1\Lambda\Pi_1 = \Lambda$, $\Pi_1 A\Pi_1 = A$, $\Pi_1 A^T\Pi_1 = A^T$, we are left with

$$\begin{cases} \partial_z\widetilde{u} = & -i\,\Xi\,A\,\widetilde{u} + \Xi\,\widetilde{v} & \text{for } \; z>0\,, \\[2mm] \partial_z\widetilde{v} = & (\Pi_1(S - i\eta_0 J^{-1})\Pi_1 - A^T\,\Xi\,A\,)\,\widetilde{u} - i\,A^T\,\Xi\,\widetilde{v} & \\ & -\Pi_1(S-i\eta_0 J^{-1})\,\Pi_0\,G(\eta_0)\Pi_0\,(S-i\eta_0 J^{-1})\,\Pi_1\,\widetilde{u} & \text{for } \; z>0\,, \\[2mm] \Pi_0 u = & -\Pi_0\,G(\eta_0)\,\Pi_0(S-i\eta_0\,J^{-1})\,\widetilde{u} & \text{for } \; z>0\,, \\[2mm] \widetilde{v} = 0 & & \text{at } z=0\,, \end{cases}$$

where $\widetilde{u} := \Pi_1 u$, $\widetilde{v} := \Pi_1 v$, Ξ denotes the inverse of $\Lambda_{|\ker(\Lambda)^\perp}$, and $G(\eta_0)$ denotes the inverse of $\Pi_0(S-i\eta_0 J^{-1})\Pi_0$ viewed as an isomorphism of $\ker\Lambda$ (hence $G(\eta_0) = \Pi_0\,G(\eta_0)$ by definition, an identity which we have used here above to write in a symmetric form the terms where it appears). Observe in particular that both Ξ and $G(\eta_0)$ are self-adjoint, like Λ and $\Pi_0(S-i\eta_0 J^{-1})\Pi_0$.

Now, choosing orthogonal bases $(C_1,\dots,C_p)$ and $(C_{p+1},\dots,C_N)$ respectively of $(\ker\Lambda)^\perp$ and $\ker\Lambda$, we get the result by taking $P = (C_1,\dots,C_p,C_{p+1},\dots,C_N)$. Indeed, in the basis $(C_1,\dots,C_N)$,

$$\Pi_0 = \left(\begin{array}{c|c} \mathbf{0}_p & \\ \hline & I_{N-p} \end{array}\right)\,,\quad \Pi_1 = \left(\begin{array}{c|c} I_p & \\ \hline & \mathbf{0}_{N-p} \end{array}\right)\,,$$

$$\Lambda = \left(\begin{array}{c|c} \breve{\Lambda} & \\ \hline & \mathbf{0}_{N-p} \end{array}\right)\,,\quad \Xi = \left(\begin{array}{c|c} \breve{\Lambda}^{-1} & \\ \hline & \mathbf{0}_{N-p} \end{array}\right)\,,$$

$$A = \left(\begin{array}{c|c} \breve{A} & \\ \hline & \mathbf{0}_{N-p} \end{array}\right)\,,\quad A^T = \left(\begin{array}{c|c} \breve{A}^T & \\ \hline & \mathbf{0}_{N-p} \end{array}\right)\,,$$

$$S - i\eta_0 J^{-1} = \left(\begin{array}{c|c} K_1(\eta_0) & K_{10}(\eta_0) \\ \hline K_{10}(\eta_0)^* & K_0(\eta_0) \end{array}\right)\,,\quad \Pi_0\,G(\eta_0)\,\Pi_0 = \left(\begin{array}{c|c} \mathbf{0}_p & \\ \hline & K_0(\eta_0)^{-1} \end{array}\right)\,,$$

where apparently empty blocks stand for null rectangular matrices, and

$$\widetilde{u} = \begin{pmatrix} \breve{u} \\ 0 \end{pmatrix}\,,\quad \widetilde{v} = \begin{pmatrix} \breve{v} \\ 0 \end{pmatrix}\,,\; u = \begin{pmatrix} \breve{u} \\ Q\,\breve{u} \end{pmatrix}$$

with

$$Q = -\,K_0(\eta_0)^{-1}\,K_{10}(\eta_0)^*\,.$$

Therefore, the differential system is of the claimed form with

$$\mathbf{H} := \left(\begin{array}{c|c} (K_{10}K_0^{-1}K_{10}^* - K_1)(\eta_0) + \breve{A}^T\,\breve{\Lambda}^{-1}\,\breve{A} & i\,\breve{A}^T\,\breve{\Lambda}^{-1} \\ \hline -\,i\,\breve{\Lambda}^{-1}\,\breve{A} & \breve{\Lambda}^{-1} \end{array}\right)\,.$$

We see in particular that $\mathbf{H}$ depends polynomially on A and A^T, and smoothly on S and on η_0 (which is meaningful since the assumption on S and η_0 is open). □

The reader will easily check that this result applies to the matrices J, Λ, $S = \Gamma_{\underline{u}} + \Sigma^{\check\eta}$, and $A = A^{\check\eta}$ involved in (22) in all three cases mentioned before, and of course that it returns the same Hamiltonian formulations of (22) as those derived separately (in the special case $p = N$, $\Pi_0 = 0$ and it suffices to replace $\Pi_0\, G(\eta_0)\,\Pi_0$, or equivalently K_0^{-1}, by zero in the expression of $\mathbf{H}$).

Now, let us say a little more about the special case (25)-(26), assuming the refined Legendre–Hadamard condition (strict rank-1 convexity): there exists $c > 0$ such that

$$\xi^2\,\check\Lambda \,+\, \xi\,(\check A^{\check\eta} + (\check A^{\check\eta})^T) \,+\, \check\Sigma^{\check\eta} \,\geq\, c\,(\,\xi^2 \,+\, \|\check\eta\|^2)\,I_n\,,\quad \forall\check\eta\in\mathbb{R}^{d-1}\,,\ \forall\xi\in\mathbb{R}\,. \tag{34}$$

Denoting by $\lambda^{\check\eta}$ the infimum for $\xi\in\mathbb{R}$ of the minimal eigenvalue of the symmetric matrix $\xi^2\,\check\Lambda \,+\, \xi\,(\check A^{\check\eta} + (\check A^{\check\eta})^T) \,+\, \check\Sigma^{\check\eta}$, we have $\lambda^{\check\eta} \geq c\|\check\eta\|^2$, and the following result due to Serre [23], which we just reformulate with our - slightly different - notations. We extend H^η to complex values of η_0, and denote for convenience by $H(\tau,\check\eta)$ the matrix $H^{(-i\tau,\check\eta)}$, defined as in (26).

Theorem 1 (D. Serre)**.** *For all $\check\eta\in\mathbb{R}^{d-1}$, for all $\tau\in\mathbb{C}$ such that $\tau^2\in\mathbb{C}\backslash(-\infty,-\lambda^{\check\eta}]$, the matrix $J\,H(\tau,\check\eta)$ is* hyperbolic *(that is, has no purely imaginary eigenvalue), and we have the following alternative*

- *either there exists $(\tau,\check\eta)$ with either $\mathsf{Re}\,\tau > 0$ or $\tau = 0$ and $\check\eta \neq 0$ such that the boundary value problem in* (25) *admits a nontrivial square integrable solution, which implies that the original boundary value problem* (8) *is* ill-posed*,*
- *or, for all $\check\eta\in\mathbb{R}^{d-1}\backslash\{0\}$, there exists $\eta_0\in\mathbb{R}$, depending smoothly on $\check\eta$, such that $\eta_0^2\in(0,\lambda^{\check\eta}]$, and for $\tau = i\eta_0$ the boundary value problem in* (25) *admits a nontrivial square integrable solution, which corresponds to a (genuine)* surface wave *solution of* (8) *if $\eta_0^2 < \lambda^{\check\eta}$.*

Note that the very first part of the statement applies in particular to complex numbers τ of positive real part. The fact that $J\,H(\tau,\check\eta)$ is hyperbolic for those τ is analogous to a well-known result due to Hersh [11], who observed that the hyperbolicity of a (first order) PDE implied the hyperbolicity of the first-order system of ODEs obtained by Fourier–Laplace transform. Rephrased directly in terms of the interior equation in (22) (that is, without reduction to a first-order system of ODEs), it says that for $\tau^2\in\mathbb{C}\backslash(-\infty,-\lambda^{\check\eta}]$, this equation has no solution of the form

$$\widehat{u}(z) = \mathrm{e}^{i\xi z}\,\widehat{u}_0\,,\quad \xi\in\mathbb{R}\,,\,\widehat{u}_0\in\mathbb{C}^N\backslash\{0\}\,. \tag{35}$$

In general, we can just show the following.

Proposition 2. *Assume that*

$$\Gamma_{\underline{u}} \,+\, \xi^2\,\Lambda \,+\, \xi\,(A^{\check\eta} + (A^{\check\eta})^T) \,+\, \Sigma^{\check\eta} \,\geq\, 0\,,\quad \forall\check\eta\in\mathbb{R}^{d-1}\,,\quad \forall\xi\in\mathbb{R}^d\,.$$

Then, when $\mathsf{Re}\,\tau > 0$, the interior equation in (22) *has no solution of the form* (35)*.*

Proof. If such a neutral mode existed, we should have

$$\tau\,\widehat{u}_0\;=\;J\,(\Gamma_{\underline{u}}\;+\;\xi^2\,\Lambda\;+\;\xi\,(A^{\check{\eta}}+(A^{\check{\eta}})^T)\;+\;\Sigma^{\check{\eta}})\,\widehat{u}_0\,,\quad \widehat{u}_0\;\neq\;0\,,$$

or in other words τ should be an eigenvalue of $J\,\mathscr{L}_{\underline{u}}(\check{\eta},\xi)$ with

$$\mathscr{L}_{\underline{u}}(\check{\eta},\xi)\;:=\;\Gamma_{\underline{u}}\;+\;\xi^2\,\Lambda\;+\;\xi\,(A^{\check{\eta}}+(A^{\check{\eta}})^T)\;+\;\Sigma^{\check{\eta}}\,.$$

By assumption, for all $(\check{\eta},\xi)\in\mathbb{R}^d$ the real symmetric matrix $\mathscr{L}_{\underline{u}}(\check{\eta},\xi)$ is nonnegative. Therefore, it does not have any negative eigenvalue. By a classical lemma (proved for completeness in appendix, see Lemma A.2), this implies that $J\,\mathscr{L}_{\underline{u}}(\check{\eta},\xi)$ does not have any eigenvalue τ of positive real part. □

Note that the main assumption in Proposition 2 is fulfilled when the Legendre–Hadamard condition in (23) holds true and when the energy E is convex with respect to u, which ensures that $\Gamma_{\underline{u}}$, the Hessian of E with respect to u at $(\underline{u},0)$, be nonnegative.

Let us now consider a situation analogous to the second case in the alternative pointed out by Serre (Theorem 1), and assume the following.

(H3) Existence of linear surface waves: There exist $\underline{\check{\eta}}\in\mathbb{R}^{d-1}\backslash\{0\}$ and $\underline{\eta}_0\in\mathbb{R}$ such that, for $\tau=i\,\underline{\eta}_0$, $\check{\eta}=\underline{\check{\eta}}$, the interior equation in (22) has no solution of the form (35), and the boundary value problem (22) has exactly a one-dimensional space of square integrable solutions.

Assume that $\widehat{u}^\eta$ is a nontrivial square integrable solution of (22) for $\tau=i\eta_0$. By reduction of the interior equations in (22) to a first-order system of ODEs coupled with a linear algebraic system (as in Proposition 1), we see that $\widehat{u}^\eta$ must go to zero exponentially fast when z goes to $+\infty$. Furthermore, by the scale invariance requirement in (H1), we find that for all $k>0$, the mapping

$$z\mapsto\widehat{r}^\eta(k,z):=\begin{pmatrix} k^{-\theta_1}\widehat{u}_1^\eta(kz)\\ \vdots\\ k^{-\theta_N}\widehat{u}_N^\eta(kz)\end{pmatrix}$$

is solution of (22) for $\tau=ik\eta_0$ and with $k\check{\eta}$ instead of $\check{\eta}$. In other words, dropping the superscript η in $\widehat{r}$ for simplicity, we have

$$i\,k\eta_0\,\widehat{r}\;=\;J\,\mathscr{L}_{\underline{u}}^{k\check{\eta}}\,\widehat{r}\;\text{ for }\;z>0\,,\quad \mathscr{C}^{k\check{\eta}}\,\widehat{r}\;=\;0\;\text{ at }z=0\,,\quad \lim_{z\to+\infty}\widehat{r}\;=\;0\,,\tag{36}$$

the limit being approached exponentially fast. Observe that by conjugation

$$-\,i\,k\eta_0\,\overline{\widehat{r}}\;=\;J\,\mathscr{L}_{\underline{u}}^{-k\check{\eta}}\,\overline{\widehat{r}}\;\text{ for }\;z>0\,,\quad \mathscr{C}^{-k\check{\eta}}\,\overline{\widehat{r}}\;=\;0\;\text{ at }z=0\,,$$

so that by setting $\widehat{r}(-k,z)=\overline{\widehat{r}(k,z)}$, we also have (36) for $k\in(-\infty,0)$. This shows that (H3) is satisfied with $k\underline{\check{\eta}}$ and $k\underline{\eta}_0$ instead of $\underline{\check{\eta}}$ and $\underline{\eta}_0$ for all $k\neq0$, which will be crucial in the derivation of weakly nonlinear surface waves.

The existence of linear surface waves is well-known in isotropic elasticity, that is in the special case (5) with $n = d$ and

$$\forall F \in \mathbb{R}^{d\times d}\,, \quad W(F) \,=\, \frac{\lambda}{2}\,(\mathrm{tr}F)^2 \,+\, \frac{\mu}{4}\,\sum_{\alpha,j}\,(F_{\alpha j} + F_{j\alpha})^2\,, \quad \mu > 0\,,\; \mu + \lambda > 0\,,$$

and they are termed after Lord Rayleigh (see the appendix for more details). As regards the fluid equations, that is with the energy as in (14), the linearized boundary value problem (16) amounts to the acoustic equations with zero normal velocity on the boundary. It is known (see for instance [5, p. 59]) to admit neutral modes, but these are not square integrable in z: in this case, what follows is irrelevant.

2.3 The amplitude equation for weakly nonlinear waves

2.3.1 Statement and comments on the amplitude equation

Let us fix $\check\eta = \underline{\check\eta} \in \mathbb{R}^{d-1}\backslash\{0\}$ and $\eta_0 = \underline{\eta}_0$ for which we have (H3). We are interested in asymptotic solutions of the fully nonlinear problem (4) of the form

$$u(t,x) = \underline{u} \,+\, \varepsilon u^{(1)}(\varepsilon t, \underline{\eta}_0 t + \underline{\check\eta}\cdot\check x, x_d) \,+\, \varepsilon^2 u^{(2)}(\varepsilon t, \underline{\eta}_0 t + \underline{\check\eta}\cdot\check x, x_d) \,+\, O(\varepsilon^3)\,, \qquad (37)$$

where both $u^{(1)}$ and $u^{(2)}$ go to zero when x_d goes to infinity. The principal part of such an expansion is called a *weakly nonlinear surface wave*: up to modulations according to the 'slow' time $s := \varepsilon t$, it propagates along the boundary with speed $\underline{\check\eta}/\underline{\eta}_0$, and decays to zero in transverse directions. Theorem 2 below says that weakly nonlinear surface waves are governed by a nonlinear amplitude equation whose (complicated) expression is in terms of the third order derivatives of E, and of $\widehat r$ defined thanks to (H3). We shall in addition discuss the properties of this equation, and show that

- it is a genuine evolution equation if and only the Lopatinskii determinant (defined in §2.4, Eq. (84)) has a simple root at $\underline{\eta}_0$ (that is, $\partial_{\eta_0}\Delta(\underline{u},\underline{\eta}) \neq 0$),
- if so then the amplitude equation reads as a nonlocal generalization of the inviscid Burgers equation, and it automatically satisfies Hunter's stability condition,
- it has a 'natural' Hamiltonian structure involving the Hilbert transform as a Hamiltonian operator.

In particular, this series of observations apply to, and sheds new light on the elasticity equations.

From now on, we assume for simplicity that $E(u,F) = E_0(u) + W(F)$, which is in particular the case for energies as in (5). This technical assumption implies that all cross-derivatives of E mixing u and F are identically zero. In particular, with the notations introduced in (1), we have $b_{\alpha\gamma m} \equiv 0$, and the scale invariance characterization in Lemma 1 reduces to the only equations in (10), (12), (13) involving the second order derivatives $\underline{a}$ and $\underline{c}$. We now need notations for the third order derivatives. Let us introduce the following

$$e_{\alpha\beta\gamma} \,:=\, \frac{\partial^3 E}{\partial u_\alpha \partial u_\beta \partial u_\gamma}\,, \quad d_{\alpha j\beta\ell\gamma m} \,:=\, \frac{\partial^3 E}{\partial F_{\alpha j}\partial F_{\beta\ell}\partial F_{\gamma m}}\,.$$

These (real) coefficients naturally arise in the third order variation of $\mathscr{E}$, which reads indeed

$$\frac{\mathrm{d}^3}{\mathrm{d}\theta^3}\,\mathscr{E}[u+\theta h]_{|\theta=0} \;=\; \int_\Omega (\mathscr{M}[u]\,(h,h))\cdot h\,\mathrm{d}x \;+\; \int_{\partial\Omega} (\mathscr{D}_{\mathbf{n}}[u]\,(h,h))\cdot h\,\mathrm{d}\check{x}\,,$$

$$(\mathscr{M}[u]\,(h,h))\cdot h' \;=\; (e_{\alpha\beta\gamma}(u)\,h_\beta\,h_\gamma \;-\; D_j(d_{\alpha j\beta\ell\gamma m}(\nabla u)\,(\partial_\ell h_\beta)(\partial_m h_\gamma))\,)\,h'_\alpha\,, \tag{38}$$

$$(\mathscr{D}_{\mathbf{n}}[u]\,(h,h))\cdot h' \;=\; d_{\alpha j\beta\ell\gamma m}(\nabla u)\,n_j\,(\partial_\ell h_\beta)(\partial_m h_\gamma)\,h'_\alpha\,. \tag{39}$$

We shall also use the notations

$$\underline{e}_{\alpha\beta\gamma} := e_{\alpha\beta\gamma}(\underline{u})\,,\quad \underline{d}_{\alpha j\beta\ell\gamma m} := d_{\alpha j\beta\ell\gamma m}(0)\,,$$

and $\mathscr{F}_y$ will mean the Fourier transform with respect to $y=\underline{\eta}_0 t+\underline{\check{\eta}}\cdot\check{x}$, with k being the Fourier variable associated with y.

Theorem 2. *Assuming (H1) and (H3), a necessary condition for* (4) *to admit a* $L^2_{x_d}$ *solution of the form* (37) *is that*

$$\mathscr{F}_y(u^{(1)}(s,y,z)) \;=\; w(s,k)\,\widehat{r}(k,z)\,,$$

where $\widehat{r}$ *satisfies* (36) *with* $\check{\eta}=\underline{\check{\eta}}$, $\eta_0=\underline{\eta}_0$ *for all* $k\neq 0$, *and the* amplitude w *(in Fourier space) is governed by the equation*

$$a_0(k)\,\partial_s w(s,k) \;+\; \int_{\mathbb{R}} a_1(k-k',k')\,w(s,k-k')\,w(s,k')\,\mathrm{d}k' \;=\; 0\,, \tag{40}$$

$$a_0(k) := \int_0^{+\infty} (J^{-1}\widehat{r}(-k,z))\cdot\widehat{r}(k,z)\,\mathrm{d}z\,, \tag{41}$$

$$\begin{aligned}
a_1(k,k') :=\; & \frac{1}{\pi}\int_0^{+\infty} \underline{e}_{\alpha\beta\gamma}\,\widehat{r}_\alpha(-k-k',z)\,\widehat{r}_\beta(k,z)\,\widehat{r}_\gamma(k',z)\,\mathrm{d}z \quad + \\
& \frac{ik\,k'\,(k+k')}{\pi}\int_0^{+\infty} \Big(\underline{d}_{\alpha j\beta\ell\gamma m}\,\eta_j\,\eta_\ell\,\eta_m\,\widehat{r}_\alpha(-k-k',z)\,\widehat{r}_\beta(k,z)\,\widehat{r}_\gamma(k',z) \\
&\quad + \underline{d}_{\alpha j\beta d\gamma m}\,\eta_j\,\eta_m\;\big(\widehat{r}_\alpha(-k-k',z)\,\widehat{r}'_\beta(k,z)\,\widehat{r}_\gamma(k',z) \\
&\qquad\qquad + \widehat{r}_\alpha(-k-k',z)\,\widehat{r}'_\beta(k',z)\,\widehat{r}_\gamma(k,z) \\
&\qquad\qquad + \widehat{r}_\alpha(k,z)\,\widehat{r}'_\beta(-k-k',z)\,\widehat{r}_\gamma(k',z)\,\big) \\
&\quad + \underline{d}_{\alpha d\beta d\gamma m}\,\eta_m\;\big(\widehat{r}'_\alpha(-k-k',z)\,\widehat{r}'_\beta(k,z)\,\widehat{r}_\gamma(k',z) \\
&\qquad\qquad + \widehat{r}'_\alpha(-k-k',z)\,\widehat{r}'_\beta(k',z)\,\widehat{r}_\gamma(k,z) \\
&\qquad\qquad + \widehat{r}'_\alpha(k',z)\,\widehat{r}'_\beta(k,z)\,\widehat{r}_\gamma(-k-k',z)\,\big) \\
&\quad + \underline{d}_{\alpha d\beta d\gamma d}\,\widehat{r}'_\alpha(-k-k',z)\,\widehat{r}'_\beta(k,z)\,\widehat{r}'_\gamma(k',z)\Big)\,\mathrm{d}z\,,
\end{aligned} \tag{42}$$

where $\widehat{r}'$ *is defined by*

$$i\,k\,\widehat{r}'(k,z) \;=\; \partial_z\widehat{r}(k,z)\,,\quad \forall k\neq 0\,,\forall z>0\,. \tag{43}$$

Let us comment on (40) before actually deriving it. To clarify the notations, let us say that the dot in the definition (41) of the coefficient $a_0(k)$ stands for the natural bilinear product in $\mathbb{C}^N$ (and not the Hilbert product), that is,

$$u \cdot v \,=\, u_\alpha\, v_\alpha \,=\, \overline{u}^*\, v \,=\, \overline{v}^*\, u\,, \quad \forall u\,,\, v \in \mathbb{C}^N\,,$$

where the bar means conjugate, the superscript $*$ means conjugate transpose, and the vectors of $\mathbb{C}^N$ are viewed as column vectors. First of all, notice that $a_0(k)$ is necessarily a *purely imaginary* number since the real matrix J^{-1} is skew-symmetric and

$$\widehat{r}(-k,z) = \overline{\widehat{r}(k,z)}\,, \quad \forall k \neq 0\,, \quad \forall z > 0\,. \tag{44}$$

It is not obvious though that $a_0(k)$ be nonzero. This question will be investigated in §2.4 by relating a_0 to the order of vanishing of the Lopatinskii determinant (which will be defined in due time).

The next question is whether, and to which extent, (40) is related to a nonlocal Burgers equation, and what are the stability/structure properties of this equation.

By the skew-symmetry of J, we have $a_0(-k) = -a_0(k) \in i\,\mathbb{R}$, and the fact that the coefficients $\underline{e}_{\alpha\beta\gamma}$ and $\underline{d}_{\alpha j\beta\ell\gamma m}$ are real together with the property in (44) imply

$$a_1(-k,-k') = \overline{a_1(k,k')}\,.$$

This is important for the amplitude equation (40) to preserve the Fourier transforms of real-valued functions. To be more precise, if $w = \mathscr{F}_y \upsilon$ with υ a real-valued function (let us say in the Schwartz class) then the inverse Fourier transform of

$$k \mapsto \frac{1}{a_0(k)} \int_{\mathbb{R}} a_1(k-k',k')\, w(s,k-k')\, w(s,k')\, \mathrm{d}k'$$

is (formally) real-valued, provided of course that a_0 does not vanish (which will be proved in Theorem 4). Furthermore, (40) is then (formally) equivalent to

$$\partial_s \upsilon \,+\, \partial_y \mathscr{Q}[\upsilon] \,=\, 0\,, \tag{45}$$

where $\upsilon = \mathscr{F}_y^{-1} w$ and $\mathscr{Q}$ is the nonlocal quadratic operator (formally) defined by

$$\mathscr{F}_y \mathscr{Q}[\upsilon](k) = \int_{\mathbb{R}} q(k-k',k')\, w(k-k')\, w(k')\, \mathrm{d}k'\,, \quad w = \mathscr{F}_y \upsilon\,, \tag{46}$$

$$q(k,k') := \frac{a_1(k,k')}{i\,(k+k')\,a_0(k+k')}\,, \quad \forall (k,k') \in \mathbf{P} \,:=\, \{(k,k')\,;\, kk'(k+k') \neq 0\}\,. \tag{47}$$

In view of the properties of a_0 and a_1, the kernel q is obviously symmetric, and such that

$$q(-k,-k') = \overline{q(k,k')}\,, \tag{48}$$

which ensures that $\mathscr{Q}$ preserve real-valued functions. We can observe that the standard, inviscid, Burgers equation is of the form in (45) with a *constant* kernel q. So we can view (45) as a non-local generalization of Burgers' equation provided that the kernel q

be bounded (a property ensuring in particular that $\mathscr{Q}$ is a continuous bilinear operator $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to H^1(\mathbb{R})$). A better situation is when q is positively homogeneous degree zero. It turns out to be the case for the kernel associated with (5) (see Remark 2 in §2.4 below), and more generally for all kernels associated with scale invariant nonlinear boundary value problems. This will be the purpose of Theorem 4 in §2.5 below.

For nonlocal Burgers equations as in (45) with q being positively homogeneous degree zero, a *stability* condition was pointed out by Hunter [12], which merely reads

$$q(-1, 0+) \;=\; q(1, 0+)\,. \tag{49}$$

The equality in (49) has been proved to be sufficient [4] and necessary [9] to get local well-posedness (in Sobolev spaces) of the Cauchy problem associated with (45). It turns out to be automatically satisfied here.

Proposition 3. *The kernel q defined in* (58) *satisfies Hunter's stability condition* (49)*.*

Proof. Recall that $a_1(k, k')$ is a symmetric function of the triplet $(k, k', -k-k')$, and that a_0 is odd. Furthermore, a_0 is continuous outside 0, and a_1 is continuous on the cone $\mathbf{P}$, that is outside the lines $k = 0$, $k' = 0$, $k + k' = 0$. Therefore, $q(k, k')$ is the product of an even function of $(k + k')$ that is continuous outside 0, and of a symmetric function of the triplet $(k, k', -k - k')$ that is continuous outside the lines $k = 0$, $k' = 0$, $k + k' = 0$. Both of them obviously take the same values at points $(-1, 0+)$ and $(1, 0+)$, and thus their product too. □

Remark 1. *Without any homogeneity property of q,* (49) *does not ensure well-posedness. In view of the proof of Lemma 2.1 in [4], what we really need is an estimate of the form*

$$|q(k, k') \,-\, q(-k-k', k')| \;\leq\; C\,\left|\frac{k'}{k}\right| \qquad \forall k, k' \in \mathbb{R}\,;\; |k'| < |k|\,.$$

Finally, we are interested in whether (45) admits a *Hamiltonian structure.* As was pointed out by Alì, Hunter and Parker [2], a sufficient condition for (45) to admit such a structure would be that q satisfy

$$q(k, k') \;=\; q(-k-k', k')\,, \quad \forall k, k' \in \mathbb{R}\,. \tag{50}$$

(This condition was actually written in a slightly different form in [2], see also [4]. Here we have already taken into account the symmetry of q and the reality condition in (48).) In this case, the Hamiltonian operator would merely be $\mathscr{J} := -\partial_y$, and the Hamiltonian functional

$$\mathscr{H}[\upsilon] \;=\; \tfrac{1}{3}\iint q(k, k')w(k)w(k')w(-k-k')\,\mathrm{d}k\,\mathrm{d}k'\,, \quad w \;=\; \mathscr{F}_y\upsilon\,. \tag{51}$$

Indeed, (50) would imply that the variational derivative of $\mathscr{H}$ at Υ is precisely $\mathscr{Q}[\upsilon]$, so that (45) would read

$$\partial_s \upsilon \;=\; \mathscr{J}\,\delta\mathscr{H}[\upsilon]\,. \tag{52}$$

However, q *does not satisfy* (50) in general. For, $a_1(k,k')$ itself is a symmetric function of the triplet $(k,k',-k-k')$, as it can readily be deduced from its definition and the symmetries in the (real) coefficients

$$\underline{e}_{\alpha\beta\gamma} = \underline{e}_{\beta\alpha\gamma} = \underline{e}_{\alpha\gamma\beta}\,, \quad \underline{d}_{\alpha j\beta\ell\gamma m} \;=\; \underline{d}_{\beta\ell\alpha j\gamma m} \;=\; \underline{d}_{\alpha j\gamma m\beta\ell}\,, \tag{53}$$

hence

$$a_1(k,k') \;=\; a_1(-k-k',k')\,, \quad \forall k,k' \in \mathbb{R}\,. \tag{54}$$

So the identity in (50) would require that $ka_0(k)$ be independent of k, which is not even true for (5), see §2.4. Nevertheless, we can play with (45) and find, under an additional assumption met in particular by (5), an equivalent equation that does have a Hamiltonian structure.

Proposition 4. *Assuming that a_0 has the following form*

$$a_0(k) \;=\; i\,\mathrm{sgn}(k)\,|k|^{-m}\,\alpha_0\,, \quad \forall k \neq 0\,, \tag{55}$$

with $\alpha_0 \in \mathbb{R}^*$ and m a rational number, Eq. (45) *is formally equivalent to*

$$\partial_s \Upsilon \,+\, \mathscr{I}\mathscr{P}[\Upsilon] \;=\; 0\,, \tag{56}$$

where $\Upsilon := |\partial_y|^{-m/2}\upsilon$, $\mathscr{I}$ denotes the Hilbert transform*, such that*

$$\Upsilon \mapsto \mathscr{I}\Upsilon\,;\; \mathscr{F}_y(\mathscr{I}\Upsilon)(k) \;=\; -\,i\,(\mathrm{sgn}k)\widehat{\Upsilon}(k)\,,$$

and $\mathscr{P}$ is the nonlocal quadratic operator (formally) defined by

$$\mathscr{F}_y\mathscr{P}[\Upsilon](k) = \int_{\mathbb{R}} p(k-k',k')\,W(k-k')\,W(k')\,\mathrm{d}k'\,, \quad W = \mathscr{F}_y\Upsilon\,, \tag{57}$$

$$p(k,k') := \frac{a_1(k,k')}{\alpha_0}\,|k+k'|^{m/2}\,|k|^{m/2}\,|k'|^{m/2}\,, \quad \forall (k,k') \in \mathbf{P}\,. \tag{58}$$

The proof is a straightforward calculation. Since the new kernel p is like a_1, symmetric in (k,k'), and symmetric in $(k,k',-k-k')$, it clearly satisfies

$$p(k,k') \;=\; p(-k-k',k')\,, \quad \forall k,k' \in \mathbb{R}\,.$$

As a consequence, Eq. (56) has the Hamiltonian structure

$$\partial_s\Upsilon \;=\; -\,\mathscr{I}\delta\mathscr{K}[\Upsilon] \tag{59}$$

with

$$\mathscr{K}[\Upsilon] \;=\; \tfrac{1}{3}\iint p(k,k')W(k)W(k')W(-k-k')\,\mathrm{d}k\,\mathrm{d}k'\,, \quad W \;=\; \mathscr{F}_y\Upsilon\,. \tag{60}$$

As will be explained in Remark 6 (see p. 34), one has (55) under some rather natural 'compatibility' condition between the matrix J and the scaling matrix $\Theta(k) := \mathrm{diag}(k^{\theta_1},\dots,k^{\theta_N})$ (namely, when $\Theta(k)$ is proportional to $J^{-1}\,\Theta(k)^{-1}\,J$). For instance for (5), we have $m = 2$ (also see the explicit computation in §2.4). Recalling that a_1 is positively homogeneous degree -1 in this special case, p is positively homogeneous degree 2, so that (56) may be viewed as a nonlocal *Hamilton–Jacobi* equation. In general, p is positively homogeneous degree $1 + m/2$.

2.3.2 Derivation of the amplitude equation

This section is devoted to the proof of Theorem 2. We seek conditions under which the fully nonlinear problem (4) admits solutions of the form

$$u(t,x) = \underline{u} \,+\, \varepsilon u^{(1)}(\varepsilon t, \underline{\eta}_0 t + \check{\underline{\eta}}\cdot\check{x}, x_d) \,+\, \varepsilon^2 u^{(2)}(\varepsilon t, \underline{\eta}_0 t + \check{\underline{\eta}}\cdot\check{x}, x_d) \,+\, O(\varepsilon^3)\,.$$

By identifying the terms of order one in ε we find that $(s,y,z)\mapsto u^{(1)}(s,y,z)$ should be a solution of the linear boundary value problem

$$\eta_0\,\partial_y u^{(1)} \;=\; J\,\mathscr{L}^{\check{\eta}}_{\underline{u}}(\partial_y,\partial_z)\,u^{(1)} \;\text{ for }\; z>0\,,\quad \mathscr{C}^{\check{\eta}}(\partial_y,\partial_z)\,u^{(1)} \;=\; 0 \;\text{ at } z=0\,, \tag{61}$$

$$\mathscr{L}^{\check{\eta}}_{\underline{u}}(\partial_y,\partial_z) \;:=\; \Gamma_{\underline{u}} \,-\, \Sigma^{\check{\eta}}\,\partial_y^2 \,-\, (A^{\check{\eta}} + (A^{\check{\eta}})^T)\,\partial_y\partial_z \,-\, \Lambda\,\partial_z^2\,. \tag{62}$$

$$\mathscr{C}^{\check{\eta}}(\partial_y,\partial_z) \;:=-\, A^{\check{\eta}}\,\partial_y \,-\, \Lambda\,\partial_z\,. \tag{63}$$

(the matrices $\Gamma_{\underline{u}}$, Λ, $A^{\check{\eta}}$, and $\Sigma^{\check{\eta}}$ being the ones defined in (19)). As to the terms of order two in ε, they yield the system

$$\eta_0\,\partial_y u^{(2)} \,-\, J\,\mathscr{L}^{\check{\eta}}_{\underline{u}}(\partial_y,\partial_z)\,u^{(2)} \;=\; -\,\partial_s u^{(1)} \,+\, \tfrac{1}{2}\,J\,\mathscr{M}^{\check{\eta}}_{\underline{u}}(\partial_y,\partial_z)(u^{(1)},u^{(1)}) \quad \text{for }\; z>0\,, \tag{64}$$

$$\mathscr{C}^{\check{\eta}}(\partial_y,\partial_z)\,u^{(2)} \;=\; -\tfrac{1}{2}\mathscr{D}^{\check{\eta}}(\partial_y,\partial_z)(u^{(1)},u^{(1)}) \qquad \text{at } z=0\,, \tag{65}$$

$$\begin{aligned}
(\mathscr{M}^{\check{\eta}}_{\underline{u}}(\partial_y,\partial_z)\,(h,h))_\alpha \;:=\;& \underline{e}_{\alpha\beta\gamma}\,h_\beta\,h_\gamma \,+ \\
&-\underline{d}_{\alpha j\beta\ell\gamma m}\,\eta_j\,\eta_\ell\,\eta_m\,\partial_y((\partial_y h_\beta)(\partial_y h_\gamma)) \,-\, \underline{d}_{\alpha d\beta\ell\gamma m}\,\eta_\ell\,\eta_m\,\partial_z((\partial_y h_\beta)(\partial_y h_\gamma)) \\
&-2\,\underline{d}_{\alpha j\beta d\gamma m}\,\eta_j\,\eta_m\,\partial_y((\partial_z h_\beta)(\partial_y h_\gamma)) \,-\, 2\,\underline{d}_{\alpha d\beta d\gamma m}\,\eta_m\,\partial_z((\partial_z h_\beta)(\partial_y h_\gamma)) \\
&-\underline{d}_{\alpha j\beta d\gamma d}\,\eta_j\,\partial_y((\partial_z h_\beta)(\partial_z h_\gamma)) \,-\, \underline{d}_{\alpha d\beta d\gamma d}\,\partial_z((\partial_z h_\beta)(\partial_z h_\gamma))\,,
\end{aligned} \tag{66}$$

$$\begin{aligned}
(\mathscr{D}^{\check{\eta}}(\partial_y,\partial_z)\,(h,h))_\alpha \;:=\;& -\,\underline{d}_{\alpha d\beta\ell\gamma m}\,\eta_\ell\,\eta_m\,(\partial_y h_\beta)(\partial_y h_\gamma) \\
&-2\,\underline{d}_{\alpha d\beta d\gamma m}\,\eta_m\,(\partial_z h_\beta)(\partial_y h_\gamma) \\
&-\underline{d}_{\alpha d\beta d\gamma d}\,(\partial_z h_\beta)(\partial_z h_\gamma)\,.
\end{aligned} \tag{67}$$

By Fourier transform in y, the first and second order problems (61) and (64)-(65) are respectively equivalent to

$$J\,\mathscr{L}^{k\check{\eta}}_{\underline{u}}\,v^{(1)} \;=\; i\,\eta_0\,k\,v^{(1)} \;\text{ for }\; z>0\,, \qquad \mathscr{C}^{k\check{\eta}}\,v^{(1)} \;=\; 0 \;\text{ at } z=0\,, \tag{68}$$

$$J\,\mathscr{L}^{k\check{\eta}}_{\underline{u}}v^{(2)} \;=\; i\,\eta_0\,k\,v^{(2)} \,+\, f \text{ for }\; z>0\,, \qquad \mathscr{C}^{k\check{\eta}}\,v^{(2)} \;=\; g \text{ at } z=0\,, \tag{69}$$

where

$$v^{(1)} \;:=\; \mathscr{F}_y u^{(1)}\,,\quad v^{(2)} \;:=\; \mathscr{F}_y u^{(2)}\,, \tag{70}$$

$$f \;:=\; \partial_s v^{(1)} \,-\, \tfrac{1}{2}\,J\,\mathscr{F}_y(\mathscr{M}^{\check{\eta}}_{\underline{u}}(\partial_y,\partial_z)(u^{(1)},u^{(1)}))\,, \tag{71}$$

$$g \;:=\; -\tfrac{1}{2}\mathscr{F}_y(\mathscr{D}^{\check{\eta}}(\partial_y,\partial_z)(u^{(1)},u^{(1)}))\,. \tag{72}$$

From the assumptions in (H1) and (H3), we know that the ODE boundary value problem in (68) has exactly a one-dimensional space of solutions tending to zero when z goes to infinity, which is spanned by $\widehat{r}$. Therefore, for any solution $v^{(1)}$ of (68) tending to zero when z goes to infinity, there exists a scalar-valued amplitude $w = w(s,k)$ such that

$v^{(1)} = w\,\widehat{r}$. Now, (69) is 'just' a nonhomogeneous version of (68). So the existence of a solution $v^{(2)}$ tending to zero when z goes to infinity is submitted to a Fredholm-type condition, which will in turn give an equation for the amplitude w. More precisely, we have the following.

Proposition 5. *The ODE boundary value problem* (69) *admits a solution* $v^{(2)}$ *tending to zero when* z *goes to infinity if and only if*

$$\int_0^{+\infty} (J^{-1}\widehat{r}(-k,z)) \cdot f(s,k,z)\,\mathrm{d}z \;=\; \widehat{r}(-k,0)\cdot g(s,k,0)\,. \tag{73}$$

Proof. The proof is based on two ingredients: 1) a duality identity; 2) the adjoint problem of (68), which turns out to admit $J^{-1}\widehat{r}(-k,z)$ as a solution.

The duality identity is

$$\int_0^{+\infty} (\mathscr{L}_{\underline{u}}^{\check{\eta}} h)\cdot h'\,\mathrm{d}z \,+\, ((\mathscr{C}^{\check{\eta}} h)\cdot h')_{|z=0} \;=\; \int_0^{+\infty} (\mathscr{L}_{\underline{u}}^{-\check{\eta}} h')\cdot h\,\mathrm{d}z \,+\, ((\mathscr{C}^{-\check{\eta}} h')\cdot h)_{|z=0}\,. \tag{74}$$

Indeed, from the expressions of $\mathscr{L}_{\underline{u}}^{\check{\eta}}$ and $\mathscr{C}^{\check{\eta}}$ in (20) and (21) respectively (see p. 10) we have

$$\int_0^{+\infty} (\mathscr{L}_{\underline{u}}^{\check{\eta}} h)\cdot h'\,\mathrm{d}z \,+\, ((\mathscr{C}^{\check{\eta}} h)\cdot h')_{|z=0} \;=$$

$$\int_0^{+\infty} (\Gamma_{\underline{u}} h \,+\, \Sigma^{\check{\eta}} h \,-\, i\,(A^{\check{\eta}} + (A^{\check{\eta}})^T)\,\partial_z h \,-\, \Lambda\,\partial_z^2 h)\cdot h'\,\mathrm{d}z \,-\, ((\,i\,A^{\check{\eta}} h \,+\, \Lambda\,\partial_z h)\cdot h')_{|z=0} \;=$$

$$\int_0^{+\infty} (\Gamma_{\underline{u}} h' \,+\, \Sigma^{\check{\eta}} h')\cdot h \,-\, (i\,(A^{\check{\eta}} + (A^{\check{\eta}})^T)\,h')\cdot\partial_z h \,-\, (\Lambda\,h')\cdot\partial_z^2 h\,\mathrm{d}z$$
$$-\,((\,i\,(A^{\check{\eta}})^T h')\cdot h \,+\, (\Lambda\,h')\cdot\partial_z h)_{|z=0}$$

by symmetry of the matrices $\Gamma_{\underline{u}}$, $\Sigma^{\check{\eta}}$, and Λ, hence (74) after integrations by parts, using that $\Sigma^{\check{\eta}}$ is quadratic in $\check{\eta}$, and that $A^{\check{\eta}}$ depends linearly on $\check{\eta}$.

Now, if (69) admits a solution $v^{(2)}$ tending to zero when z goes to infinity, taking the inner product of the interior equation with h, another function tending to zero when z goes to infinity, we get

$$\int_0^{+\infty} (i\,\eta_0\,k\,v^{(2)} \,+\, f)\cdot h\,\mathrm{d}z \;=\; \int_0^{+\infty} (J\,\mathscr{L}_{\underline{u}}^{k\check{\eta}} v^{(2)})\cdot h\,\mathrm{d}z = \;-\int_0^{+\infty} (\mathscr{L}_{\underline{u}}^{k\check{\eta}} v^{(2)})\cdot(Jh)\,\mathrm{d}z$$

by the skew-symmetry of J. Using the duality identity (74) and the boundary equation in (69), this yields

$$\int_0^{+\infty} (i\,\eta_0\,k\,v^{(2)} + f)\cdot h\,\mathrm{d}z \;=\; -\int_0^{+\infty} (\mathscr{L}_{\underline{u}}^{-k\check{\eta}} Jh)\cdot v^{(2)}\,\mathrm{d}z - ((\mathscr{C}^{-\check{\eta}} Jh)\cdot v^{(2)})_{|z=0} + (g\cdot Jh)_{|z=0}\,,$$

which finally reduces to

$$\int_0^{+\infty} f\cdot h\,\mathrm{d}z \;=\; (g\cdot Jh)_{|z=0}$$

provided that h is solution to the *adjoint problem*

$$\mathscr{L}_{\underline{u}}^{-k\check{\eta}}\, Jh \;=\; -\,i\,\eta_0\,k\,h \;\text{ for }\; z>0\,, \quad \mathscr{C}^{-k\check{\eta}}\, Jh \;=\; 0 \;\text{ at } z=0\,. \tag{75}$$

To conclude, we observe that $h(z) := J^{-1}\widehat{r}(-k,z)$ is solution of this problem, by definition of $\widehat{r}$. □

Proof of Theorem 2. Recalling the definitions of f and g from (71) and (72) respectively, we deduce from (74) that

$$a_0(k)\,\partial_s w \;+\; \frac{1}{2}\int_0^{+\infty} \widehat{r}(-k,z))\cdot\, \mathscr{F}_y(\mathscr{M}_{\underline{u}}^{\check{\eta}}(\partial_y,\partial_z)(u^{(1)},u^{(1)})))\,\mathrm{d}z \;=\; -\,\frac{1}{2}\,\widehat{r}(-k,0)\cdot\mathscr{F}_y(\mathscr{D}^{\check{\eta}}(\partial_y,\partial_z)(u^{(1)},u^{(1)}))_{|z=0}\,, \tag{76}$$

where we have used again the skew-symmetry of J to simplify the second term,

$$a_0(k) \;:=\; \int_0^{+\infty} (J^{-1}\widehat{r}(-k,z))\cdot\widehat{r}(k,z)\,\mathrm{d}z\,,$$

and $\mathscr{F}_y u^{(1)} \;=\; w\,\widehat{r}$. We shall finally arrive at the announced equation (40) for w by using the basic formula $2\pi\,\mathscr{F}(u\times v) \;=\; \mathscr{F}(u)*\mathscr{F}(v)$. Before that, it is important to notice that the boundary terms in the right-hand side of (76) here above cancel out after integration by parts of the corresponding terms in the second integral in the left-hand side. Equation (77) thus becomes

$$a_0(k)\,\partial_s w \;+\; \tfrac{1}{2}\int_0^{+\infty} (\widehat{r}_\alpha(-k,z)\,\mathscr{F}_y\,N_\alpha \;+\; (\partial_z\widehat{r}_\alpha(-k,z))\,\mathscr{F}_y P_\alpha)\,\mathrm{d}z \;=\; 0\,, \tag{77}$$

$$\begin{aligned} N_\alpha \;:=\;\; & \underline{e}_{\alpha\beta\gamma}\,u_\beta^{(1)}\,u_\gamma^{(1)} \;-\; \underline{d}_{\alpha j\beta\ell\gamma m}\,\eta_j\,\eta_\ell\,\eta_m\,\partial_y((\partial_y u_\beta^{(1)})(\partial_y u_\gamma^{(1)}))\\ & -2\,\underline{d}_{\alpha j\beta d\gamma m}\,\eta_j\,\eta_m\,\partial_y((\partial_z u_\beta^{(1)})(\partial_y u_\gamma^{(1)})) \;-\; \underline{d}_{\alpha j\beta d\gamma d}\,\eta_j\,\partial_y((\partial_z u_\beta^{(1)})(\partial_z u_\gamma^{(1)}))\,, \end{aligned}$$

$$\begin{aligned} P_\alpha \;:=\; & \underline{d}_{\alpha d\beta\ell\gamma m}\,\eta_\ell\,\eta_m\,(\partial_y u_\beta^{(1)})(\partial_y u_\gamma^{(1)}) \;+\; 2\,\underline{d}_{\alpha d\beta d\gamma m}\,\eta_m\,(\partial_z u_\beta^{(1)})(\partial_y u_\gamma^{(1)})\\ & + \underline{d}_{\alpha d\beta d\gamma d}\,(\partial_z u_\beta^{(1)})(\partial_z u_\gamma^{(1)})\,, \end{aligned}$$

so that

$$\begin{aligned} 2\pi\,\mathscr{F}_y N_\alpha \;=\; & \underline{e}_{\alpha\beta\gamma}\,(w\,\widehat{r}_\beta)*(w\,\widehat{r}_\gamma) \;+\; i\,\underline{d}_{\alpha j\beta\ell\gamma m}\,\eta_j\,\eta_\ell\,\eta_m\,k(kw\,\widehat{r}_\beta)*(kw\,\widehat{r}_\gamma)\\ & +\;2\,\underline{d}_{\alpha j\beta d\gamma m}\,\eta_j\,\eta_m\,k\,(w\,\partial_z\widehat{r}_\beta)*(kw\,\widehat{r}_\gamma) \;-\; i\,\underline{d}_{\alpha j\beta d\gamma d}\,\eta_j\,k\,(w\,\partial_z\widehat{r}_\beta)*(w\,\partial_z\widehat{r}_\gamma)\,,\\ 2\pi\,\mathscr{F}_y P_\alpha \;=\; & -\;\underline{d}_{\alpha d\beta\ell\gamma m}\,\eta_\ell\,\eta_m\,(kw\,\widehat{r}_\beta)*(kw\,\widehat{r}_\gamma) \;+\; 2i\,\underline{d}_{\alpha d\beta d\gamma m}\,\eta_m\,(w\,\partial_z\widehat{r}_\beta)*(kw\,\widehat{r}_\gamma)\\ & +\;\underline{d}_{\alpha d\beta d\gamma d}\,(w\,\partial_z\widehat{r}_\beta)*(w\,\partial_z\widehat{r}_\gamma)\,. \end{aligned}$$

Therefore, (77) equivalently reads

$$a_0(k)\,\partial_s w \;+\; \int_{\mathbb{R}} a_1(k-k',k')\,w(k-k')\,w(k')\,\mathrm{d}k' \;=\; 0\,,$$

with

$$a_1(k,k') := \frac{1}{\pi}\int_0^{+\infty} \varrho(k,k',z)\,\mathrm{d}z\,,$$

$$\begin{aligned}
\varrho(k,k') := \;& \big(\underline{e}_{\alpha\beta\gamma} + i\,\underline{d}_{\alpha j\beta\ell\gamma m}\,\eta_j\,\eta_\ell\,\eta_m\,k\,k'\,(k+k')\big)\,\widehat{r}_\alpha(-k-k')\,\widehat{r}_\beta(k)\,\widehat{r}_\gamma(k') \\
& + 2\,\underline{d}_{\alpha j\beta d\gamma m}\,\eta_j\,\eta_m\,k'\,(k+k')\,\widehat{r}_\alpha(-k-k')\,(\partial_z\widehat{r}_\beta(k))\,\widehat{r}_\gamma(k') \\
& - i\,\underline{d}_{\alpha j\beta d\gamma d}\,\eta_j\,(k+k')\,\widehat{r}_\alpha(-k-k')\,(\partial_z\widehat{r}_\beta(k))\,(\partial_z\widehat{r}_\gamma(k')) \\
& - \underline{d}_{\alpha d\beta\ell\gamma m}\,\eta_\ell\,\eta_m\,k\,k'\,(\partial_z\widehat{r}_\alpha(-k-k'))\,\widehat{r}_\beta(k)\,\widehat{r}_\gamma(k') \\
& + 2i\,\underline{d}_{\alpha d\beta d\gamma m}\,\eta_m\,k'\,(\partial_z\widehat{r}_\alpha(-k-k'))\,(\partial_z\widehat{r}_\beta(k))\,\widehat{r}_\gamma(k') \\
& + \underline{d}_{\alpha d\beta d\gamma d}\,(\partial_z\widehat{r}_\alpha(-k-k'))\,(\partial_z\widehat{r}_\beta(k))\,(\partial_z\widehat{r}_\gamma(k'))\,.
\end{aligned}$$

We have omitted to write dependencies on z in the definition of ϱ for simplicity. In fact, it is preferable to rewrite (77) as

$$a_0(k)\,\partial_s w + \int_{\mathbb{R}} a_1^{\mathrm{sym}}(k-k',k')\,w(k-k')\,w(k')\,\mathrm{d}k' = 0\,,$$

with the symmetrized kernel $a_1^{\mathrm{sym}} = \frac{1}{\pi}\int_0^{+\infty}\varrho^{\mathrm{sym}}\mathrm{d}z$,

$$\begin{aligned}
\varrho^{\mathrm{sym}}(k,k') := \;& \big(\underline{e}_{\alpha\beta\gamma} + i\,\underline{d}_{\alpha j\beta\ell\gamma m}\,\eta_j\,\eta_\ell\,\eta_m\,k\,k'\,(k+k')\big)\,\widehat{r}_\alpha(-k-k')\,\widehat{r}_\beta(k)\,\widehat{r}_\gamma(k') \\
& + \underline{d}_{\alpha j\beta d\gamma m}\,\eta_j\,\eta_m\;\big(\,k'\,(k+k')\,\widehat{r}_\alpha(-k-k')\,(\partial_z\widehat{r}_\beta(k))\,\widehat{r}_\gamma(k') \\
& \qquad + k\,(k+k')\,\widehat{r}_\alpha(-k-k')\,(\partial_z\widehat{r}_\beta(k'))\,\widehat{r}_\gamma(k) \\
& \qquad - k\,k'\,\widehat{r}_\alpha(k)\,(\partial_z\widehat{r}_\beta(-k-k'))\,\widehat{r}_\gamma(k')\,\big) \\
& + i\,\underline{d}_{\alpha d\beta d\gamma m}\,\eta_m\;\big(\,k'\,(\partial_z\widehat{r}_\alpha(-k-k'))\,(\partial_z\widehat{r}_\beta(k))\,\widehat{r}_\gamma(k') \\
& \qquad + k\,(\partial_z\widehat{r}_\alpha(-k-k'))\,(\partial_z\widehat{r}_\beta(k'))\,\widehat{r}_\gamma(k) \\
& \qquad - (k+k')\,(\partial_z\widehat{r}_\alpha(k'))\,(\partial_z\widehat{r}_\beta(k))\,\widehat{r}_\gamma(-k-k')\,\big) \\
& + \underline{d}_{\alpha d\beta d\gamma d}\,(\partial_z\widehat{r}_\alpha(-k-k'))\,(\partial_z\widehat{r}_\beta(k))\,(\partial_z\widehat{r}_\gamma(k'))\,.
\end{aligned}$$

In order to 'simplify' the definition of ϱ^{sym} here above, we have used here the symmetry of the coefficients

$$\underline{d}_{\alpha j\beta\ell\gamma m} = \underline{d}_{\beta\ell\alpha j\gamma m} = \underline{d}_{\alpha j\gamma m\beta\ell}\,.$$

□

2.4 Evolutionarity of the amplitude equation

This section is focussed on the coefficient a_0 of the time derivative in (40). For convenience, let us recall its definition from (41),

$$a_0(k) := \int_0^{+\infty}(J^{-1}\widehat{r}(-k,z))\cdot\widehat{r}(k,z)\,\mathrm{d}z\,,\quad k\neq 0\,,$$

where $\widehat{r}$ is a nontrivial element of the one-dimensional space of solutions of (36) for $\check{\eta} = \underline{\check{\eta}}$, $\eta_0 = \underline{\eta}_0$,

$$ik\,\underline{\eta}_0\,\widehat{r} = J\,\mathscr{L}_{\underline{u}}^{k\underline{\check{\eta}}}\,\widehat{r}\;\text{ for }\; z>0\,,\quad \mathscr{C}^{k\underline{\check{\eta}}}\,\widehat{r} = 0\;\text{ at } z=0\,,\quad \lim_{z\to+\infty}\widehat{r} = 0\,. \tag{78}$$

Our aim is to point out a necessary and sufficient condition on the linearized BVP problem (22) for $a_0(k)$ to be non zero for all $k \neq 0$.

As a warm-up, let us start with an explicit computation of a_0 when the energy is as in (5). In this case, the boundary value problem in (78) amounts to finding

$$\widehat{r}(k,z) \;=\; \begin{pmatrix} k^{-1}\widehat{\rho}(kz) \\ i\,\underline{\eta}_0\widehat{\rho}(kz) \end{pmatrix} \quad \text{for } k>0\,,$$

where $\widehat{\rho}$ is such that

$$\begin{cases} \underline{\eta}_0^2\,\widehat{\rho}(z) \;=\; (\check{\Sigma}^{\check{\underline{\eta}}} \;-\; i\,(\check{A}^{\check{\underline{\eta}}} + (\check{A}^{\check{\underline{\eta}}})^T)\,\partial_z \;-\; \check{\Lambda}\,\partial_z^2)\,\widehat{\rho}(z) \;\text{ for }\; z>0\,, \\ (i\,\check{A}^{\check{\underline{\eta}}} \;+\; \check{\Lambda}\,\partial_z)\,\widehat{\rho} \;=\; 0 \;\text{ at }\; z=0\,, \\ \lim_{z\to+\infty}\widehat{\rho}(z) \;=\; 0\,. \end{cases} \tag{79}$$

To fix the ideas, let us look for instance at (the well-known case of) isotropic elasticity with Lamé coefficients μ and λ such that $\mu>0$ and $\mu+\lambda>0$. We find (see the appendix for more details) that any solution $\widehat{\rho}$ of (79) must be of the form

$$\widehat{\rho}(z) \;=\; \mathrm{e}^{-\omega_1 z}\,\rho_1 \;+\; \mathrm{e}^{-\omega_2 z}\,\rho_2\,,\quad \omega_1 \;:=\; \sqrt{\|\check{\underline{\eta}}\|^2 - \underline{\eta}_0^2/\mu}\,,\quad \omega_2 \;:=\; \sqrt{\|\check{\underline{\eta}}\|^2 - \underline{\eta}_0^2/(2\mu+\lambda)}\,,$$

where $\underline{\eta}_0$ is solution to the homogeneous equation

$$\left(\eta_0^2/(2\mu) \;-\; \|\check{\underline{\eta}}\|^2\right)^2 \;-\; \|\check{\underline{\eta}}\|^2\,\sqrt{\|\check{\underline{\eta}}\|^2 - \eta_0^2/\mu}\,\sqrt{\|\check{\underline{\eta}}\|^2 - \eta_0^2/(2\mu+\lambda)} \;=\; 0\,, \tag{80}$$

and

$$\rho_{1,2} \;=\; \begin{pmatrix} \zeta_{1,2} \\ \sigma_{1,2} \end{pmatrix}\,,$$

where $\zeta_{1,2}\in\mathbb{C}^{d-1}$, $\sigma_{1,2}\in\mathbb{C}$ are such that

$$\check{\underline{\eta}}\cdot\zeta_1 \;+\; i\,\sigma_1\omega_1 \;=\; 0\,,\quad \begin{pmatrix} \zeta_2 \\ \sigma_2 \end{pmatrix} \parallel \begin{pmatrix} \check{\underline{\eta}} \\ i\,\omega_2 \end{pmatrix}\,. \tag{81}$$

In addition, we see by (81) that $\rho_1\cdot\overline{\rho_2} \;=\; i\,\sigma_1\,(\omega_1-\omega_2)$, and without loss of generality (see the appendix for an explanation), we can choose σ_1 to be real, hence $\mathsf{Re}(\rho_1\cdot\overline{\rho_2}) \;=\; 0$. This in turn gives

$$(J^{-1}\widehat{r}(-k,z))\cdot\widehat{r}(k,z) \;=\; 2\,ik^{-1}\underline{\eta}_0\,\|\widehat{\rho}(kz)\|^2 \;=\; 2\,ik^{-1}\underline{\eta}_0\,\left(\mathrm{e}^{-2\omega_1 kz}\,\|\rho_1\|^2 \;+\; \mathrm{e}^{-2\omega_2 kz}\,\|\rho_2\|^2\right)\,,$$

and finally

$$a_0(k) \;=\; ik^{-2}\eta_0\left(\frac{\|\rho_1\|^2}{\omega_1} \;+\; \frac{\|\rho_2\|^2}{\omega_2}\right) \;\neq\; 0 \;\text{ for }\; k>0\,.$$

Observe in passing that a_0 is positively homogeneous degree -2 in k. We shall come back to this property in Section 2.5.

Remark 2. *On the example in* (5)*, we also easily obtain homogeneity for* a_1*. Indeed, the special form*

$$E(\chi,p)=\frac{1}{2}\|p\|^2+W(\nabla\chi)$$

implies that $e_{\alpha\beta\gamma}\equiv 0$ *for all* α*,* β*,* γ*, and* $d_{\alpha j\beta\ell\gamma m}\equiv 0$ *as soon as either one of* α*,* β *or* γ *is greater than* n*. Therefore, the kernel* ϱ *defined in the proof of Theorem 2 (see p. 24) involves only the components of* $\widehat{r}(\ell,z)$ *that are, up to combinations of exponentials* $\mathrm{e}^{-\omega_{1,2}\ell z}$*, positively homogeneous degree* -1 *in* ℓ *for* $\ell\in\{k,k',-k-k'\}$*. This together with* $\underline{e}_{\alpha\beta\gamma}=0$ *shows that* ϱ *is a combination of those exponentials times positively homogeneous degree zero (*i.e. $3-3$*) terms. After integration in* z *we 'loose' one degree, so in turn* a_1 *is positively homogeneous degree* -1*. This implies that the mapping*

$$(k,k')\mapsto\frac{a_1(k,k')}{a_0(k+k')}$$

is positively homogeneous degree one. As shown in Theorem 4 in Section 2.5 below, this homogeneity property is a general fact for the kernel associated with scale invariant nonlinear boundary value problems (4)*.*

The fact that $a_0(k)$ be nonzero for $k\neq 0$ may look rather mysterious at this stage, and possibly specific to the example above. In fact, as already observed by Marcou [15] for first-order boundary value problems with maximal dissipative boundary conditions, the nonvanishing of a_0 is directly linked to the fact that the Lopatinskii determinant has a simple root at $\tau=i\,\underline{\eta}_0$. It is now time to define the Lopatinskii determinant for the general boundary value problem in (22) with $\tau=i\eta_0$. In this respect we need a reformulation in terms of a first-order system of ODEs, which can be done thanks to Proposition 1. We shall assume that this proposition (see p. 12) applies to $\eta_0=\underline{\eta}_0$, and to the matrices $S=\Gamma_{\underline{u}}+\Sigma^{\check{\underline{\eta}}}$, $A=A^{\check{\underline{\eta}}}$, Λ defined in (19). Note that the only assumption concerning η_0 in Proposition 1 is (30), which is open with respect to η_0 (and also $\check{\eta}$ but we shall not use it). So it is automatically satisfied for η in a neighborhood of $\underline{\eta}$. Therefore, there exists a $(2p)\times(2p)$ Hermitian matrix $\mathbf{H}^{\eta}_{\underline{u}}$ depending smoothly on $\eta=(\eta_0,\check{\underline{\eta}})$ for η_0 in a neighborhood of $\underline{\eta}_0$, such that there is a one-to-one correspondence between solutions to

$$\left\{\begin{array}{ll} i\eta_0\,\widehat{u}\;=\;J\,(\Gamma_{\underline{u}}\,+\,\Sigma^{\check{\eta}})\,\widehat{u}\;-\;i\,J\,(A^{\check{\eta}}+(A^{\check{\eta}})^T)\,\partial_z\widehat{u}\;-\;J\,\Lambda\,\partial_z^2\widehat{u} & \text{for}\ \ z>0\,,\\ i\,A^{\check{\eta}}\widehat{u}\;+\;\Lambda\,\partial_z\widehat{u}\;=\;0_N & \text{at}\ z=0\,,\\ \lim_{z\to+\infty}\widehat{u}\;=\;0_N\,, & \end{array}\right. \tag{82}$$

and solutions to

$$\partial_z\begin{pmatrix}\breve{u}\\ \breve{v}\end{pmatrix}\;=\;\mathbf{J}\,\mathbf{H}^{\eta}_{\underline{u}}\begin{pmatrix}\breve{u}\\ \breve{v}\end{pmatrix}\ \ \text{for}\ \ z>0\,,\quad \breve{v}_{|z=0}\;=\;0_p\,,\quad \lim_{z\to+\infty}\begin{pmatrix}\breve{u}\\ \breve{v}\end{pmatrix}=\begin{pmatrix}0_p\\ 0_p\end{pmatrix}. \tag{83}$$

We recall that the matrix $\mathbf{J}$ above is just

$$\mathbf{J}\;=\;\begin{pmatrix}\mathbf{0}_p & I_p\\ -I_p & \mathbf{0}_p\end{pmatrix}.$$

By the first assumption in (H3), the matrix $\mathbf{J}\,\mathbf{H}^{\underline{\eta}}_{\underline{u}}$ has to be hyperbolic, and since this is an open property, it is valid for $\mathbf{J}\,\mathbf{H}^{\eta}_{\underline{u}}$ with η in a neighborhood of $\underline{\eta}$. Thus the second part of (H3) can equivalently be stated in terms of the stable subspace $\mathbb{E}^s(\underline{u}, \underline{\eta})$ of $\mathbf{J}\,\mathbf{H}^{\underline{\eta}}_{\underline{u}}$. It requires indeed that for $\eta = \underline{\eta}$, (83) have exactly a one-dimensional space of square integrable solutions, or equivalently that the intersection of $\mathbb{E}^s(\underline{u}, \underline{\eta})$ with $\mathbb{C}^p \times \{0_p\}$ be a line. The latter condition is easily coined into the algebraic condition that the $p \times p$ matrix $(\Pi\mathbf{R}_1(\underline{u}, \underline{\eta}), \dots, \Pi\mathbf{R}_p(\underline{u}, \underline{\eta}))$ be of rank $p-1$, where Π denotes the projection

$$\begin{array}{cccc} \Pi : & \mathbb{C}^{2p} & \to & \mathbb{C}^p \\ & \begin{pmatrix} \check{u} \\ \check{v} \end{pmatrix} & \mapsto & \check{v}\,, \end{array}$$

and $(\mathbf{R}_1(\underline{u}, \eta), \dots, \mathbf{R}_p(\underline{u}, \eta))$ denotes a basis of $\mathbb{E}^s(\underline{u}, \eta)$. The fact that $\mathbb{E}^s(\underline{u}, \eta)$ is necessarily of dimension p comes from the observation that the eigenvalues of $\mathbf{J}\,\mathbf{H}^{\eta}_{\underline{u}}$ are exactly pairwise, the pairs being of the form[7] $(-\omega, \overline{\omega})$ with ω of positive real part. This implies in particular that the stable and unstable subspaces of $\mathbf{J}\,\mathbf{H}^{\eta}_{\underline{u}}$, which we denote respectively by $\mathbb{E}^s(\underline{u}, \eta)$ and $\mathbb{E}^u(\underline{u}, \eta)$, are both of dimension p (half the size of $\mathbf{J}\,\mathbf{H}^{\eta}_{\underline{u}}$). From a more analytical point of view, the algebraic condition above requires that the *Lopatinskii determinant*

$$\Delta(\underline{u}, \eta) \;:=\; \det(\Pi\mathbf{R}_1(\underline{u}, \eta), \dots, \Pi\mathbf{R}_p(\underline{u}, \eta)) \tag{84}$$

vanish at $\eta = \underline{\eta}$.

Remark 3. *If the mapping $\eta_0 \mapsto \Delta(\underline{u}, \eta_0, \check{\underline{\eta}})$ vanishes at order one at $\eta_0 = \underline{\eta}_0$, that is,*

$$\partial_{\eta_0}\Delta(\underline{u}, \underline{\eta}_0, \check{\underline{\eta}}) \;\neq\; 0\,, \tag{85}$$

then necessarily the matrix $(\Pi\mathbf{R}_1(\underline{u}, \underline{\eta}), \dots, \Pi\mathbf{R}_p(\underline{u}, \underline{\eta}))$ is of rank $(p-1)$, but the converse is not true (see Lemma A.5). Thus we will have to assume (85) *in complement to (H3) when we want to prove that a_0 is nonzero.*

Theorem 3. *We assume (H1), (H3), and that*

$$\ker\Lambda \subset \ker A^{\check{\underline{\eta}}}\,, \quad \ker\Lambda \subset \ker(A^{\check{\underline{\eta}}})^T\,,$$

$$\Pi_0(\Gamma_{\underline{u}} + \Sigma^{\check{\underline{\eta}}} - i\,\underline{\eta}_0 J^{-1})\Pi_0 \;\; \textit{is an isomorphism of } \ker\Lambda\,,$$

where Π_0 denotes the orthogonal projector onto the kernel of Λ in $\mathbb{C}^N$. For $\eta = (\eta_0, \check{\underline{\eta}})$ with η_0 close enough to $\underline{\eta}_0$, we denote by $\mathbf{H}^{\eta}_{\underline{u}}$ the Hermitian matrix given by Proposition 1 (on p. 12), which has a block structure of the form

$$\mathbf{H}^{\eta}_{\underline{u}} \;=\; \left(\begin{array}{c|c} K(\underline{u}, \eta) & M(\check{\eta}) \\ \hline M(\check{\eta})^* & H \end{array}\right),$$

where $K(\underline{u}, \eta)$ and H are $p \times p$ Hermitian matrices, and assume that the hyperbolic matrix $\mathbf{J}\mathbf{H}^{\eta}_{\underline{u}}$ is diagonalizable. Then the coefficient a_0 as defined in (41) *is such that*

$$a_0(1) \;=\; i\,\partial_{\eta_0}\Delta(\underline{u}, \underline{\eta})\,,$$

where Δ is the Lopatinskii determinant defined as in (84) *by means of a basis $(\mathbf{R}_1(\underline{u}, \eta), \dots, \mathbf{R}_p(\underline{u}, \eta))$ of $\mathbb{E}^s(\underline{u}, \eta)$, the stable subspace of $\mathbf{J}\mathbf{H}^{\eta}_{\underline{u}}$.*

[7] Because the adjoint matrix $(\mathbf{J}\,\mathbf{H}^{\eta}_{\underline{u}})^* = -\mathbf{H}^{\eta}_{\underline{u}}\,\mathbf{J}$ is conjugated to $-\mathbf{J}\,\mathbf{H}^{\eta}_{\underline{u}}$.

Proof. In order to make the connection between a_0 and $\partial_{\eta_0}\Delta$, let us derive some more explicit formulae for these quantities.

1) Computation of $\partial_{\eta_0}\Delta$. The way, if not the detailed computation, is rather classical. Since $\mathbf{J}\,\mathbf{H}^{\eta}_{\underline{u}}$ is assumed to be diagonalizable, $\mathbb{E}^s(\underline{u},\eta)$ and $\mathbb{E}^u(\underline{u},\eta)$ admit bases made of (genuine) eigenvectors. Let us denote by $(\mathbf{R}_\alpha)_{\alpha\in\{1,\dots,p\}}$ an independent family of eigenvectors of $\mathbf{J}\,\mathbf{H}^{\eta}_{\underline{u}}$ associated with (not necessarily distinct) eigenvalues $(-\omega_\alpha)_{\alpha\in\{1,\dots,p\}}$ of negative real parts. This makes a basis of $\mathbb{E}^s(\underline{u},\eta)$. For simplicity, we have omitted to write the dependence of $\mathbf{R}_\alpha$ on $\underline{u},\eta$. We shall specify when these vectors are evaluated at $\underline{\eta}$ by underlining them. Similarly, we denote by $(\mathbf{R}_{p+\alpha})_{\alpha\in\{1,\dots,p\}}$ a basis of $\mathbb{E}^u(\underline{u},\eta)$ made of eigenvectors of $\mathbf{J}\,\mathbf{H}^{\eta}_{\underline{u}}$ associated with the eigenvalues $(-\omega_{p+\alpha} := \overline{\omega}_\alpha)_{\alpha\in\{1,\dots,p\}}$, of negative real parts. We also introduce the basis $(\mathbf{L}_\alpha)_{\alpha\in\{1,\dots,2p\}}$ of $\mathbb{C}^{2p}$ defined by

$$\mathbf{L}_\alpha^*\,\mathbf{J}\,\mathbf{R}_\beta \;=\; \delta_{\alpha\beta}\,,\quad \forall\alpha\,,\,\beta\in\{1,\dots,2p\}\,. \tag{86}$$

By an easy computation (see Lemma A.3) we see that $\mathbf{L}_\alpha$ must be an eigenvector of $\mathbf{J}\,\mathbf{H}^{\eta}_{\underline{u}}$ associated with $\overline{\omega}_\alpha \;=\; -\,\omega_{p+\alpha}$ (this equality being true for all $\alpha\in\{1,\dots,2p\}$ under the natural convention that $\omega_\beta \;=\; \omega_{\beta-2p}$ for $\beta\in\{2p+1,\dots,3p\}$). If the eigenvalues are distinct, up to a renormalization of the eigenvectors, we merely have $\mathbf{L}_\alpha \;=\; \mathbf{R}_{p+\alpha}$ (again with the convention that $\mathbf{R}_\beta \;=\; \mathbf{R}_{\beta-2p}$ for $\beta\in\{2p+1,\dots,3p\}$). For convenience, we introduce the additional notations

$$\mathbf{L}_\alpha \;=\; \begin{pmatrix} N_\alpha \\ L_\alpha \end{pmatrix},\quad \mathbf{R}_\alpha \;=\; \begin{pmatrix} R_\alpha \\ S_\alpha \end{pmatrix},\quad N_\alpha\,,\; L_\alpha\,,\; R_\alpha\,,\; S_\alpha\,\in\mathbb{C}^p\,.$$

These vectors will be underlined when evaluated at $\underline{\eta}$.

We consider Δ defined as in (84), that is, $\Delta = \det(S_1,\dots,S_p)$. By (H3), the matrix $(\underline{S}_1,\dots,\underline{S}_p)$ is of rank $(p-1)$. This implies[8] in particular that

$$\begin{array}{ccc} \mathbb{C}^p & \to & \mathbb{C} \\ V & \mapsto & \det(V,\underline{S}_2,\dots,\underline{S}_p) \end{array}$$

is a nontrivial linear form. Thus there exists a unique $\underline{L}\in\mathbb{C}^p$ such that

$$\forall V\in\mathbb{C}^p\,,\;\underline{L}^*V \;=\; \det(V,\underline{S}_2,\dots,\underline{S}_p)\,.$$

An alternate way of formulating (H3) is that there exists $(\lambda_1,\dots,\lambda_p)\in\mathbb{C}^p\backslash\{(0,\dots,0)\}$ such that

$$\sum_{\alpha=1}^{p}\lambda_\alpha\,\underline{S}_\alpha = 0\,.$$

Without loss of generality we can assume that $\lambda_1 = 1$, and by the same trick as in Lemma A.5 we readily see that

$$\partial_{\eta_0}\Delta(\underline{u},\,\underline{\eta}) \;=\; \sum_{\alpha=1}^{p}\lambda_\alpha\,\det(\partial_{\eta_0}S_\alpha(\underline{u},\,\underline{\eta}),\underline{S}_2,\dots,\underline{S}_p) \;=\; \sum_{\alpha=1}^{p}\lambda_\alpha\,\underline{L}^*\,\partial_{\eta_0}S_\alpha(\underline{u},\,\underline{\eta})$$

[8] For simplicity, we assume that $\underline{S}_1$ belongs to the vector space spanned by $\underline{S}_2,\dots,\underline{S}_p$. This is always possible up to reordering the indeces.

by definition of $\underline{L}$. Now, if we decompose

$$\partial_{\eta_0}\mathbf{R}_\alpha \;=\; \mu_{\alpha\beta}\mathbf{R}_\beta\,,$$

we have, using that $\underline{L}^*\underline{S}_\beta \;=\; 0$ for all $\beta \in \{1,\ldots,p\}$,

$$\partial_{\eta_0}\Delta(\underline{u},\underline{\eta}) \;=\; \sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{2p}\lambda_\alpha\;\underline{\mu}_{\,\alpha\beta}\,\underline{L}^*\,\underline{S}_\beta\,.$$

It thus remains to obtain an 'explicit' expression for $\underline{\mu}_{\,\alpha\beta}$. This is done by differentiating

$$\mathbf{H}^\eta\mathbf{R}_\alpha \;-\; \omega_\alpha\,\mathbf{J}\,\mathbf{R}_\alpha \;=\; 0$$

with respect to η_0. By using (86) we get rid of the term with a derivative of ω_α, and thus obtain

$$\mu_{\alpha\beta} \;=\; \frac{\mathbf{L}_\beta^*\,(\partial_{\eta_0}\mathbf{H}^\eta)\,\mathbf{R}_\alpha}{\omega_\alpha-\omega_\beta}\quad\forall\alpha\,,\,\beta\,;\;\omega_\alpha\neq\omega_\beta\,.$$

Note that ω_α and ω_β are always distinct if $\alpha\in\{1,\ldots,p\}$ and $\beta\in\{p+1,\ldots,2p\}$ (the former being of negative real part and the latter of positive real part). Finally, using the structure of $\mathbf{H}^\eta$, we have $\mathbf{L}_\beta^*\,(\partial_{\eta_0}\mathbf{H}^\eta)\,\mathbf{R}_\alpha \;=\; N_\beta^*\,(\partial_{\eta_0}K)\,R_\alpha$, hence

$$\underline{\mu}_{\,\alpha\beta} \;=\; \frac{\underline{N}_\beta^*\,(\partial_{\eta_0}K)(\underline{\eta})\,\underline{R}_\alpha}{\underline{\omega}_\alpha-\underline{\omega}_\beta}\,,$$

and finally,

$$\partial_{\eta_0}\Delta(\underline{u},\underline{\eta}) \;=\; \sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{2p}\frac{\lambda_\alpha\,(\underline{N}_\beta^*\,(\partial_{\eta_0}K)(\underline{\eta})\,\underline{R}_\alpha)\,(\underline{L}^*\,\underline{S}_\beta)}{\underline{\omega}_\alpha-\underline{\omega}_\beta}\,.$$

2) Computation of a_0. By definition, $\widehat{r}(1,\cdot)$ spans the (one-dimensional) space of solutions of (82), and the link between the solutions of (82) and the solutions of (83) has been made precise in Proposition 1. It shows in particular that

$$\widehat{r}^*J^{-1}\widehat{r} \;=\; \begin{pmatrix}\check{r}\\ Q\check{r}\end{pmatrix}^*P^*J^{-1}P\begin{pmatrix}\check{r}\\ Q\check{r}\end{pmatrix},$$

with $Q \;=\; -\,K_0(\eta)^{-1}\,K_{10}(\eta)^*$, where the blocks K_0, K_{10}, and K_1 are those found in

$$P^*\,(\Gamma_{\underline{u}}+\Sigma^{\check{\eta}}\;-\;i\eta_0\,J^{-1})\,P \;=\; \left(\begin{array}{c|c} K_1(\eta) & K_{10}(\eta)\\ \hline K_{10}(\eta)^* & K_0(\eta)\end{array}\right).$$

We obviously have that

$$-\,i\,P^*J^{-1}P \;=\; \left(\begin{array}{c|c} \partial_{\eta_0}K_1(\eta) & \partial_{\eta_0}K_{10}(\eta)\\ \hline \partial_{\eta_0}K_{10}(\eta)^* & \partial_{\eta_0}K_0(\eta)\end{array}\right).$$

Furthermore, from Proposition 1 we know that the left-upper block in $\mathbf{H}^\eta_{\underline{u}}$ is given by

$$K(\eta) \;:=\; -\,K_1(\eta)\;+\;K_{10}(\eta)\,K_0(\eta)^{-1}\,K_{10}(\eta)^*\;+\;B(\check{\eta})\,,$$

where $B(\breve{\eta}) = (\breve{A}^{\breve{\eta}})^T \breve{\Lambda} \breve{A}^{\breve{\eta}}$ does not depend on η_0. We thus see after a little calculation that

$$i \begin{pmatrix} \breve{r} \\ Q\breve{r} \end{pmatrix}^* P^* J^{-1} P \begin{pmatrix} \breve{r} \\ Q\breve{r} \end{pmatrix} = \breve{r}^* (\partial_{\eta_0} K) \, \breve{r} \, .$$

Now, $\breve{r}$ is related to the algebraic objects defined above by

$$\breve{r}(1, z) = \sum_{\alpha=1}^{p} \lambda_\alpha \, \mathrm{e}^{-\omega_\alpha z} \, \underline{R}_\alpha \, ,$$

so that

$$i \, a_0(1) = - \int_0^{+\infty} \breve{r}(1, z)^* (\partial_{\eta_0} K) \, \breve{r}(1, z) \, \mathrm{d}z = - \sum_{\alpha=1}^{p} \sum_{\gamma=1}^{p} \frac{\overline{\lambda}_\gamma \, \lambda_\alpha \, \underline{R}_\gamma^* \, (\partial_{\eta_0} K) \, \underline{R}_\alpha}{\overline{\omega}_\gamma + \omega_\alpha} \, .$$

3) Conclusion We have $\overline{\omega}_\gamma = -\omega_{p+\gamma}$, and we can assume without loss of generality that $\mathbf{L}_{p+\gamma} = \mathbf{R}_\gamma$, which implies in particular that $N_{p+\gamma} = R_\gamma$. Furthermore, a bit of algebra (see Lemma A.4) shows that

$$\overline{\lambda}_\gamma = \underline{L}^* \, \underline{S}_{p+\gamma} \, .$$

This eventually shows that

$$i \, a_0(1) = \partial_{\eta_0} \Delta(\underline{u}, \underline{\eta}) \, .$$

$\square$

Remark 4. *Applying Theorem 3 at point $k\underline{\eta}$ instead of $\underline{\eta}$, we also get*

$$i \, a_0(k) = k^{-1} \, \partial_{\eta_0} \Delta(\underline{u}, k\underline{\eta}) \, .$$

It is not clear yet why $a_0(1) \neq 0$ should imply $a_0(k) \neq 0$ for all $k \neq 0$. Nevertheless, we claim this is true, and shall prove it in the next section by scale invariance arguments.

2.5 Back to scale invariance and homogeneity properties

The purpose of this section is to investigate further the relationships between $\Delta(\underline{u}, k\underline{\eta})$ and $\Delta(\underline{u}, \underline{\eta})$, between $a_0(1)$ and $a_0(k)$, and to show *in fine* the homogeneity of the kernel q.

There are two ways of interpreting the scale invariance assumption in (H1). One is mostly algebraic and follows from (10) and (12) in Lemma 1. The former implies that for all $k \neq 0$, for all α, β,

$$J_{\alpha\gamma} \, \underline{a}_{\gamma\beta} \, k^{-1+\theta_\alpha-\theta_\beta} = J_{\alpha\gamma} \, \underline{a}_{\gamma\beta} \, ,$$

which can be written with matrices as

$$\Theta(k) \, J \, \Gamma_{\underline{u}} \, \Theta(k)^{-1} = k \, J \, \Gamma_{\underline{u}} \, ,$$

where $\Theta(k)$ is the diagonal matrix $\mathrm{diag}(k^{\theta_1}, \ldots, k^{\theta_N})$, whose inverse is obviously $\Theta(1/k)$. Similarly, (12) implies that for all $k \neq 0$, for all α, β, j, ℓ,

$$J_{\alpha\gamma} \, \underline{c}_{\gamma j \beta \ell} \, k^{1+\theta_\alpha-\theta_\beta} = J_{\alpha\gamma} \, \underline{c}_{\gamma j \beta \ell} \, ,$$

hence the following algebraic identities,

$$\Theta(k)\, J\, \Sigma^{k\check{\eta}}\, \Theta(k)^{-1} \,=\, k\, J\, \Sigma^{\check{\eta}}\,, \quad \Theta(k)\, J\, \Lambda\, \Theta(k)^{-1} \,=\, k^{-1}\, J\, \Lambda\,, \tag{87}$$

$$\Theta(k)\, J\, A^{k\check{\eta}}\, \Theta(k)^{-1} \,=\, J\, A^{\check{\eta}}\,, \quad \Theta(k)\, J\, (A^{k\check{\eta}})^T\, \Theta(k)^{-1} \,=\, J\, (A^{\check{\eta}})^T\,, \tag{88}$$

where we have also used that $\Sigma^{\check{\eta}}$ is quadratic in $\check{\eta}$ and that $A^{\check{\eta}}$ depends linearly on $\check{\eta}$. In particular, these identities imply the following ones,

$$\Theta(k)\, J\, (\Gamma_{\underline{u}} \,+\, \Sigma^{k\check{\eta}} \,-\, i\eta_0 k\, J^{-1})\, \Theta(k)^{-1} \,=\, k\, J\, (\Gamma_{\underline{u}} \,+\, \Sigma^{\check{\eta}} \,-\, i\eta_0 J^{-1})\,, \tag{89}$$

$$\Theta(k)\, J\, (ik\omega\, (A^{k\check{\eta}} \,+\, (A^{k\check{\eta}})^T) \,-\, (k\omega)^2\Lambda)\, \Theta(k)^{-1} \,=\, k\, J\, (i\omega\, (A^{\check{\eta}} \,+\, (A^{\check{\eta}})^T) \,-\, \omega^2\Lambda)\,. \tag{90}$$

The analytical way of viewing (H1) is the one we have used to define $\widehat{r}(k,z)$. Namely, there is a one-to-one correspondence between solutions to (22) for $\tau = i\eta_0$ and solutions of (22) with $\check{\eta}$ replaced by $k\check{\eta}$ and $\tau = ik\eta_0$. This correspondence reads

$$\Theta(k)^{-1}\, \widehat{u}^{\eta}(kz) \,=\, \widehat{u}^{k\eta}(z)\,, \tag{91}$$

with the obvious notation $\widehat{u}^{\eta}$ for solutions of (22) with $\tau = i\eta_0$. In terms of $\widehat{r}$, (91) reads

$$\widehat{r}(k,z) \,=\, \Theta(k)^{-1}\, \widehat{r}(1,kz)\,, \quad \forall k > 0\,. \tag{92}$$

These observations will be used in a crucial way in the proof of the following.

Theorem 4. *Under the assumptions, and with the notations of Theorems 2 and 3, if* $\partial_{\eta_0}\Delta(\underline{u}, \underline{\eta}\,) \neq 0$ *then we have* $a_0(k) \neq 0$ *for all* $k \neq 0$*. Moreover, if we make the reinforced assumption*

(H1♯) Nonlinear scale invariance: *The boundary value problem* (4) *is invariant by the rescaling*

$$(x, t, u_1, \ldots, u_N) \mapsto (kx, kt, k^{\theta_1} u_1, \ldots, k^{\theta_N} u_N)\,,$$

for all $k > 0$*,*

then the mapping

$$(k, k') \mapsto \frac{a_1(k,k')}{a_0(k+k')}$$

is positively homogeneous degree one, and the kernel q *in* (46) *is thus positively homogeneous degree zero.*

Proof. The first statement follows from Theorem 3, Remark 4, and Proposition 6 stated and proved below, which altogether show that

$$a_0(k) \,=\, i\, k^{-1}\, D(k)\, \partial_{\eta_0}\Delta(\underline{u}, \underline{\eta}) \,=\, k^{-1}\, D(k)\, a_0(1) \,\neq\, 0\,.$$

For the second part, let us recall that by its definition in (41),

$$a_0(k) \,=\, \int_0^{+\infty} h(k,z) \cdot \widehat{r}(k,z)\, \mathrm{d}z\,, \quad k \neq 0\,,$$

where h is solution to the adjoint problem in (75) for $\eta = \underline{\eta}$. Equivalently, we have that $h(k,z) = \overline{h(-k,z)}$, where $h(-k,\cdot)$ is solution to

$$\mathscr{L}_{\underline{u}}^{k\check{\underline{\eta}}}\, Jh \;=\; i\,\underline{\eta}_0\, k\, h \;\text{ for }\; z > 0\,, \quad \mathscr{C}^{k\check{\underline{\eta}}}\, Jh \;=\; 0 \;\text{ at } z = 0\,. \tag{93}$$

The key point to show the homogeneity of a_0 lies in the counterpart for h of (92) for $\widehat{r}$. It is easily seen from the algebraic identities in (87)-(87) that (93) is invariant under the change of scale associated with the matrix $\Theta(k)^{-1}$ when the direct problem

$$J\,\mathscr{L}_{\underline{u}}^{k\check{\underline{\eta}}} u \;=\; i\,\eta_0\, k\, u \;\text{ for }\; z > 0\,, \quad \mathscr{C}^{k\check{\underline{\eta}}}\, u \;=\; 0 \;\text{ at } z = 0 \tag{94}$$

is invariant under the change of scale associated with $\Theta(k)$. Moreover, for all $k \neq 0$, there is an obvious one-to-one correspondence between solutions of (93) and of (94), which reads $Jh = u$. Therefore, there exists a mapping $k \neq 0 \mapsto \lambda(k) \neq 0$ such that

$$h(-k,z) = \lambda(k)\,\Theta(k)\,h(-1,kz)\,, \quad \forall k > 0\,. \tag{95}$$

The (analogous) identities in (92) and (95) show that

$$a_0(k) \;=\; \overline{\lambda(k)} \int_0^{+\infty} \Theta(k)\,h(1,kz)\cdot\Theta(k)^{-1}\,\widehat{r}(1,kz)\,\mathrm{d}z \;=\; \frac{\overline{\lambda(k)}}{k}\, a_0(1)$$

by a straightforward change of variable (we have just used here that $\Theta(k)$ is symmetric). This gives an alternate, and in fact more direct way of justifying that

$$a_0(1) \neq 0 \quad \Leftrightarrow \quad (a_0(k) \neq 0\,,\ \forall k \neq 0)\,.$$

We now turn to the properties of a_1. The scale invariance of the nonlinear problem requires that both the first-order and second-order systems (61) and (64) be invariant by the rescaling

$$(y, z, s, \eta, u^{(1)}, u^{(2)}) \mapsto (y, kz, ks, \eta/k, \Theta(k)u^{(1)}, \Theta(k)u^{(2)})\,,$$

which implies that the differential operators defined in (62) and (66) satisfy, for all $k \neq 0$,

$$\Theta(k)\,J\,\mathscr{L}_{\underline{u}}^{k\check{\eta}}(\partial_y, k\partial_z)\,\Theta(k)^{-1} \;=\; k\,J\,\mathscr{L}_{\underline{u}}^{\check{\eta}}(\partial_y,\partial_z)\,,$$

$$\Theta(k)\,J\,\mathscr{M}_{\underline{u}}^{k\check{\eta}}(\partial_y, k\partial_z)\,(\Theta(k)^{-1}u, \Theta(k)^{-1}u) \;=\; k\,J\,\mathscr{M}_{\underline{u}}^{\check{\eta}}(\partial_y,\partial_z)(u,u)\,.$$

The former equality leads to the algebraic conditions (10)-(12) in Lemma 1. Similarly, the latter yields the identities

$$(\theta_\varepsilon - \theta_\beta - \theta_\gamma - 1)\,J_{\varepsilon\alpha}\,\underline{e}_{\alpha\beta\gamma} \;=\; 0\,, \tag{96}$$

$$(\theta_\varepsilon - \theta_\beta - \theta_\gamma + 2)\;J_{\varepsilon\alpha}\,\underline{d}_{\alpha j\beta\ell\gamma m} \;=\; 0\,. \tag{97}$$

Let us examine for instance the first term in $a_1(k,k')$ (see p. 24), the other ones being analogous. Up to a factor π, it is the integral in z over $(0,+\infty)$ of

$$\varrho_0(k,k',z) := \underline{e}_{\alpha\beta\gamma}\,\widehat{r}_\alpha(-k-k',z)\,\widehat{r}_\beta(k,z)\,\widehat{r}_\gamma(k',z) \;=\; \underline{e}_{\alpha\beta\gamma}\,J_{\alpha\varepsilon}\,h_\varepsilon(k+k',z)\,\widehat{r}_\beta(k,z)\,\widehat{r}_\gamma(k',z)$$

$$= \overline{\lambda(k+k')}\,(k+k')^{\theta_\varepsilon}\,k^{-\theta_\beta}\,(k')^{-\theta_\gamma}\,\underline{e}_{\alpha\beta\gamma}\,J_{\alpha\varepsilon}\,h_\varepsilon(1,(k+k')z)\,\widehat{r}_\beta(1,kz)\,\widehat{r}_\gamma(1,k'z)\,.$$

Therefore, using (96) and the skew-symmetry of J, we see that for all $\nu > 0$,

$$\varrho_0(\nu k, \nu k', z) \,=\, \nu\,\frac{\overline{\lambda(\nu(k+k'))}}{\overline{\lambda(k+k')}}\,\varrho_0(k,k',\nu\,z) \,=\, \nu^2\,\frac{a_0(\nu(k+k'))}{a_0(k+k')}\,\varrho_0(k,k',\nu\,z)$$

After integration in z, we loose one degree in ν. $\square$

Remark 5. *Unlike* (10)-(12)*, the identities* (96)-(97) *are not invariant by the addition of a constant to the* θ_α*'s. This reflects the nonlinearity of the underlying problem. Nevertheless, one can easily check that* (96)-(97) *are satisfied in the special case* (5) *with* $\theta_\alpha = 1$ *for* $\alpha \leq n$ *and* $\theta_\alpha = 0$ *for* $\alpha \geq n+1$.

Proposition 6. *Under the assumptions, and with the notations of Theorem 3, there exists a rational function* $D = D(k)$ *such that*

$$\Delta(\underline{u}, k\,\underline{\eta}) \,=\, D(k)\,\Delta(\underline{u},\,\underline{\eta}) \quad \forall k > 0\,.$$

Proof. If we forget for a while the boundary condition in (22), we also have that any solution of the form

$$\widehat{u}^{\eta}(z) \,=\, \mathrm{e}^{-\omega(\eta) z}\,U(\eta)$$

of the interior equations in (22) with $\tau = i\eta_0$ is associated with a solution of the interior equations in (22) with $\check{\eta}$ replaced by $k\check{\eta}$ and $\tau = ik\eta_0$, $k > 0$, namely

$$\widehat{u}^{k\eta}(z) \,:=\, \mathrm{e}^{-\omega(k\eta) z}\,U(k\eta)\,, \quad \omega(k\eta) := k\omega(\eta)\,, \quad U(k\eta) \,:=\, \Theta(k)^{-1}\,U(\eta)\,.$$

(This is easily checked by using (89)-(90).) As a consequence, we can make a one-to-one correspondence between eigenvectors of $\mathbf{J}\mathbf{H}^{\eta}_{\underline{u}}$ and those of $\mathbf{J}\mathbf{H}^{k\eta}_{\underline{u}}$. Indeed, let $(\mathbf{R}_\alpha(\eta))_{\alpha\in\{1,\dots,N\}}$ be, as in the previous section, a basis of eigenvectors of $\mathbf{J}\mathbf{H}^{\eta}_{\underline{u}}$,

$$\mathbf{R}_\alpha(\eta) \,=\, \begin{pmatrix} R_\alpha(\eta) \\ S_\alpha(\eta) \end{pmatrix}$$

being associated with the eigenvalue $\omega_\alpha(\eta)$. We know that

$$P^*\,(i\,A^{\check{\eta}} \,-\, \omega_\alpha(\eta)\,\Lambda)\,U_\alpha(\eta) \,=\, \begin{pmatrix} S_\alpha(\eta) \\ 0_{N-p} \end{pmatrix}\,, \quad U_\alpha(\eta) \,:=\, P\,\begin{pmatrix} R_\alpha(\eta) \\ Q(\eta)\,R_\alpha(\eta) \end{pmatrix}\,,$$

where P and $Q(\eta)$ are defined as in Proposition 1. Then we get a basis $(\mathbf{R}_\alpha(k\eta))_{\alpha\in\{1,\dots,N\}}$ of eigenvectors of $\mathbf{J}\mathbf{H}^{k\eta}_{\underline{u}}$ associated with the eigenvalues $\omega_\alpha(k\eta) = k\omega_\alpha(\eta)$ by setting

$$\mathbf{R}_\alpha(k\eta) \,=\, \begin{pmatrix} R_\alpha(k\eta) \\ S_\alpha(k\eta) \end{pmatrix}\,,$$

$$R_\alpha(k\eta) \,:=\, \begin{pmatrix} I_p \,\big|\, & 0_{p\times(N-p)} \end{pmatrix}\,P^*\,U_\alpha(k\eta)\,, \quad U_\alpha(k\eta) := \Theta(k)^{-1}\,U_\alpha(\eta)\,,$$

$$S_\alpha(k\eta) \,:=\, \begin{pmatrix} I_p \,\big|\, & 0_{p\times(N-p)} \end{pmatrix}\,P^*\,(i\,A^{k\check{\eta}} \,-\, k\omega_\alpha(\eta)\,\Lambda)\,U_\alpha(k\eta)\,.$$

(This choice does not preserve the normalization property $\mathbf{R}^*_{p+\alpha}\,\mathbf{J}\,\mathbf{R}_\beta \,=\, \delta_{\alpha\beta}$ but this is not important here.) By definition, we have

$$\Delta(\underline{u},k\,\underline{\eta}) \,=\, \det(S_1(k\eta),\ldots,S_p(k\eta)) \,=\, \det\left(\begin{array}{c|c} S_1(k\eta)\ \cdots\ S_p(k\eta) & \\ \hline & I_{N-p} \end{array}\right)$$

$$= \det(P^*(i\,A^{k\check{\eta}} - k\omega_1(\eta)\,\Lambda)\,U_1(k\eta),\ldots,P^*(i\,A^{k\check{\eta}} - k\omega_p(\eta)\,\Lambda)\,U_p(k\eta),P^*C_{p+1},\ldots,P^*C_N)\,.$$

(Remind that the C_α's are the column vectors of the unitary matrix P.) Therefore, $\Delta(\underline{u},k\,\underline{\eta}) \,=$

$$\det((i\,A^{k\check{\eta}} \,-\, k\omega_1(\eta)\,\Lambda)\,\Theta(k)^{-1}\,U_1(\eta),\ldots,(i\,A^{k\check{\eta}} \,-\, k\omega_p(\eta)\,\Lambda)\,\Theta(k)^{-1}\,U_p(\eta),C_{p+1},\ldots,C_N)$$

$$\begin{aligned} =\ & \det(\Theta(k)J)^{-1}\times \\ & \det(J\,(i\,A^{\check{\eta}} \,-\, \omega_1(\eta)\,\Lambda)\,U_1(\eta),\ldots,J\,(i\,A^{\check{\eta}} \,-\, \omega_p(\eta)\,\Lambda)\,U_p(\eta),\Theta(k)JC_{p+1},\ldots,\Theta(k)JC_N) \end{aligned}$$

by (87)-(88). This in turn gives

$$\begin{aligned} & \Delta(\underline{u},k\,\underline{\eta}) \,=\, \det\Theta(k)^{-1}\times \\ & \det((i\,A^{\check{\eta}} \,-\, \omega_1(\eta)\,\Lambda)\,U_1(\eta),\ldots,(i\,A^{\check{\eta}} \,-\, \omega_p(\eta)\,\Lambda)\,U_p(\eta),J^{-1}\Theta(k)JC_{p+1},\ldots,J^{-1}\Theta(k)JC_N) \\ & =\, D(k)\times\Delta(\underline{u},\,\underline{\eta}) \end{aligned}$$

with

$$D(k) \,:=\, \det\Theta(k)^{-1}\,\det\big(\left(\begin{array}{c|c} 0_{(N-p)\times p} & I_{N-p}\end{array}\right)\,(P^*J^{-1}\Theta(k)JC_{p+1},\ldots,P^*J^{-1}\Theta(k)JC_N)\big)\,.$$

From a purely algebraic point of view, it is not clear why $D(k)$ should be nonzero (unless we have an identity as in (98) below). However, by the one-to-one correspondence in (91), we know that $\Delta(\underline{u},k\,\underline{\eta})$ cannot vanish for any $k\neq 0$ if $\Delta(\underline{u},\,\underline{\eta})$ does not. □

Remark 6. *Assume that $J^{-1}\,\Theta(k)^{-1}$ be proportional to $\Theta(k)^{-1}\,J^{-1}$, which is a trivial way of ensuring the compatibility of* (92) *and* (95) *with $Jh(-k,z) = \widehat{r}(k,z)$, and is equivalent to $J^{-1}\,\Theta(k)^{-1}\,J$ being proportional to $\Theta(k)$. By inspection of the eigenvalues of these matrices, we see that this requires*

$$\{k^{-\theta_1},\ldots,k^{-\theta_N}\} \,=\, \lambda(k)\,\{k^{\theta_1},\ldots,k^{\theta_N}\}$$

for some real-valued mapping $k>0\mapsto\lambda(k)>0$. Equivalently, the set $\{1,\ldots,N\}$ splits into $A\cup B$ in such a way that for all $\alpha\in A$, $\beta\in B$,

$$\theta_\alpha+\theta_\beta\,+\,\theta\,=\,0\,,$$

and $\lambda(k)\,=\,k^\theta$, for some rational number θ. This gives rise to the identity

$$\Theta(k)^{-1}\,J^{-1}\,\Theta(k)^{-1} \,=\, k^\theta\,J^{-1}\,, \tag{98}$$

and incidentally shows that

$$a_0(k) \,=\,=\, k^{\theta-1}\,a_0(1)$$

is positively homogeneous degree $\theta-1$. Eq. (98) *happens to be true in particular with $\theta=-1$ for the system in* (5)*, since $\Theta(k)\,=\,\mathrm{diag}(k\,,\ldots,k,1,\ldots,1)$, which confirms - as already found by calculation - that a_0 is positively homogeneous degree -2 in this case.*

3 Boundary value problems for first order systems in fixed domains

3.1 General framework and main result

In this section, we are interested in weakly nonlinear surface waves for first order hyperbolic systems in a fixed domain. For the sake of simplicity, we shall restrict as in the previous section to the case when the spatial domain is the half-space $\Omega = \{x \in \mathbb{R}^d \, ; \; x_d > 0\}$. We consider a homogeneous initial-boundary value problem (IBVP)

$$\begin{cases} A_0(u)\, \partial_t u + \sum_{j=1}^d A_j(u)\, \partial_j u = 0 \, , & x \in \Omega \, , t > 0 \, , \\ B(u)\, u = 0 \, , & x \in \partial\Omega \, , t > 0 \, , \\ u_{|t=0} = u_0 \, , & x \in \Omega \, . \end{cases} \tag{99}$$

In all this section, we assume that N is an integer, $\mathscr{O}$ is a neighborhood of the origin in $\mathbb{R}^{2N}$, the unknown u in (99) belongs to $\mathbb{R}^{2N}$, the matrix-valued functions $A_0, \dots, A_d$ are smooth (at least $\mathscr{C}^1$) on $\mathscr{O}$ with values in $\mathbb{R}^{2N \times 2N}$, and B is a smooth (at least $\mathscr{C}^1$) function on $\mathscr{O}$ with values in $\mathbb{R}^{N \times 2N}$. The reason why we choose matrices with these specific sizes will be explained below. We first assume that the operator in (99) is hyperbolic with constant multiplicity in the time direction.

Assumption 1. *The matrix $A_0(u)$ is invertible for all $u \in \mathscr{O}$ and there exist an integer $q \geq 1$, some real functions $\lambda_1, \dots, \lambda_q$ that are $\mathscr{C}^1$ on $\mathscr{O} \times (\mathbb{R}^d \setminus \{0\})$, analytic and positively homogeneous of degree 1 with respect to their second argument, and there also exist some positive integers $\nu_1, \dots, \nu_q$ such that*

$$\forall\, u \in \mathscr{O} \, , \forall\, \xi \in \mathbb{R}^d \setminus \{0\} \, , \quad \det \Big[\tau\, I + \sum_{j=1}^d \xi_j\, A_0(u)^{-1}\, A_j(u) \Big] = \prod_{k=1}^q \big(\tau + \lambda_k(u, \xi) \big)^{\nu_k} \, .$$

Moreover the eigenvalues $\lambda_1(u, \xi), \dots, \lambda_q(u, \xi)$ of $\sum_j \xi_j\, A_0(u)^{-1}\, A_j(u)$ are assumed to be semi-simple *(which means that their algebraic multiplicity ν_k equals their geometric multiplicity) and such that*

$$\lambda_1(u, \xi) < \cdots < \lambda_q(u, \xi) \, , \quad \forall u \in \mathscr{O} \, , \; \forall \xi \in \mathbb{R}^d \setminus \{0\} \, .$$

For simplicity we also assume that the boundary $\partial\Omega$ is non-characteristic for the hyperbolic operator.

Assumption 2. *The matrix $A_d(u)$ is invertible for all $u \in \mathscr{O}$, and $A_0(u)^{-1}\, A_d(u)$ has N positive as well as N negative eigenvalues, counted with multiplicity. Moreover, the matrix $B(u)$ has maximal rank N for all $u \in \mathscr{O}$.*

Up to restricting the neighborhood $\mathscr{O}$, we have the following standard result.

Lemma 2. *The set $\mathscr{M} := \{u \in \mathscr{O} \, , \, B(u)\, u = 0\}$ is a $\mathscr{C}^1$ submanifold of $\mathbb{R}^{2N}$ of dimension N, whose tangent space $T_0\mathscr{M}$ at 0 coincides with* $\ker B(0)$*.*

There is a one-to-one correspondence between the constant solutions to (99) and the elements of $\mathscr{M}$. Given $u \in \mathscr{M}$, the linearized IBVP around u reads

$$\begin{cases} A_0(u)\,\partial_t v + \sum_{j=1}^d A_j(u)\,\partial_j v = 0\,, & x \in \Omega\,, t > 0\,, \\ (\mathrm{d}B(u)\cdot v)\,u + B(u)\,v = 0\,, & x \in \partial\Omega\,, t > 0\,, \\ v_{|t=0} = v_0\,, & x \in \Omega\,. \end{cases} \tag{100}$$

Let us observe that there is no obstacle in considering (100) also when u does not belong to $\mathscr{M}$, but of course such an IBVP would not describe the approximate evolution for small perturbations of a particular solution to (99).

Following [14], see also [8, chapter 4] for a detailed description, we may deduce the well-posedness of (100) from some spectral properties derived by means of the Laplace-Fourier transform (Laplace in time and Fourier in the direction of $\partial\Omega$). For future use, we thus introduce the notation

$$\forall\,(u,\tau,\check{\eta}) \in \mathscr{O} \times \mathbb{C} \times \mathbb{R}^{d-1}\,, \quad A(u,\tau,\check{\eta}) := \tau\,A_0(u) + \sum_{j=1}^{d-1} \eta_j\,A_j(u)\,. \tag{101}$$

When $\tau = \eta_0$ belongs to $\mathbb{R}$, we shall feel free to use the notation $\eta := (\tau,\check{\eta}) \in \mathbb{R}^d$, $A(u,\eta)$ for $A(u,\eta_0,\check{\eta})$, and similarly for other symbols. The following result dates back to the original work by Hersh [11], and has been subsequently improved by Kreiss [14] and Métivier [16].

Theorem 5. *Let Assumptions 1 and 2 be satisfied. Then for all $u \in \mathscr{O}$, $\tau \in \mathbb{C}$ of negative imaginary part and $\check{\eta} \in \mathbb{R}^{d-1}$, the matrix $-i\,A_d(u)^{-1}\,A(u,\tau,\check{\eta})$ is hyperbolic and its stable subspace $\mathbb{E}^s(u,\tau,\check{\eta})$ has dimension N. Moreover, $\mathbb{E}^s$ extends as a continuous bundle over $\mathscr{O} \times \{(\tau,\check{\eta}) \in \mathbb{C} \times \mathbb{R}^{d-1}\,,\ \mathsf{Im}\,\tau \le 0\,,\ (\tau,\check{\eta}) \neq (0,0)\}$.*

We are now going to make a crucial assumption about the spectral stability of (100), which is analogous to the second case in the alternative of Theorem 1.

Assumption 3. *For all $u \in \mathscr{M}$, $\tau \in \mathbb{C}$ of negative imaginary part and $\check{\eta} \in \mathbb{R}^{d-1}$, the only solution to*

$$X \in \mathbb{E}^s(u,\tau,\check{\eta})\,, \quad (\mathrm{d}B(u)\cdot X)\,u + B(u)\,X = 0\,,$$

is 0. Moreover, there exists $\underline{\eta} \in \mathbb{R}^d \setminus \{0\}$ such that $-i\,A_d(0)^{-1}\,A(0,\underline{\eta})$ is hyperbolic and $\mathbb{E}^s(0,\underline{\eta}) \cap \ker B(0)$ is not reduced to $\{0\}$.

Before going further, let us comment on the two parts of Assumption 3. The first one means that the linearized problem (100) about any constant solution u of (99) sufficiently close to 0 satisfies the Lopatinskii condition. If it were not the case, there would exist a sequence (u_n) in $\mathscr{M}$ converging to 0 and such that the Lopatinskii condition be violated for each linearized problem around u_n. Then there would basically be no hope for a stability/existence result for (99) in the framework of Sobolev spaces. Indeed, very small perturbations of 0 would trigger one of the states u_n and an arbitrarily high frequency, and the resulting mode would be exponentially amplified due to the failure of the Lopatinskii condition. So the first part of Assumption 3 is very reasonable if we expect (99) to be well-posed even in any weak sense. In the second part of Assumption 3, we consider the case when the linearized problem around 0 does not satisfy the uniform Lopatinskii condition, due to the existence of surface waves. Moreover, assuming that $-i\, A_d(0)^{-1} A(0,\underline{\eta})$ be hyperbolic amounts to restricting the failure of the uniform Lopatinskii condition to the elliptic frequency region. Assumption 3 already appeared in earlier work devoted to neutrally stable nonlinear hyperbolic boundary value problems, see in particular [22, 15]. Incidentally, let us observe that the hyperbolicity of the matrix $-i\, A_d(0)^{-1} A(0,\eta)$ implies that the size of the matrices $A_j(u)$ is even and that the number of stable eigenvalues equals half the size of these matrices. This justifies our choice to denote by $2N$ the size of $A_j(u)$ and $B(u)$.

Remark 7. *By homogeneity of A with respect to η, Assumption 3 also holds true at $k\eta$ for all $k \in \mathbb{R}$. However, this plays no role in what follows.*

Our final assumption specifies the behavior of the symbol A and the degeneracy of the Lopatinskii condition at frequency $\underline{\eta}$.

Assumption 4. *There exist a neighborhood $\mathscr{V}$ of $\underline{\eta}$ in $\mathbb{C} \times \mathbb{R}^{d-1}$ and a $\mathscr{C}^1$ mapping*

$$\begin{array}{rccc} T: & \mathscr{O} \times \mathscr{V} & \rightarrow & Gl_{2N}(\mathbb{C}) \\ & (u,\tau,\check{\eta}) & \mapsto & T(u,\tau,\check{\eta})\,, \end{array}$$

holomorphic in τ, and such that

$$\begin{aligned} T(u,\tau,\check{\eta})^{-1} & \left(-i\, A_d(u)^{-1}\, A(u,\tau,\check{\eta})\right) T(u,\tau,\check{\eta}) \\ & = \operatorname{diag}\left(\beta_1^-(u,\tau,\check{\eta}),\dots,\beta_N^-(u,\tau,\check{\eta}),\beta_1^+(u,\tau,\check{\eta}),\dots,\beta_N^+(u,\tau,\check{\eta})\right), \end{aligned}$$

where for all $(u,\tau,\check{\eta}) \in \mathscr{O} \times \mathscr{V}$, there holds $\pm \,\mathsf{Re}\, \beta_n^{\pm}(u,\tau,\check{\eta}) > 0$, $n \in \{1,\dots,N\}$. Furthermore, denoting by $R_n^-(u,\tau,\check{\eta})$, $n = 1,\dots,N$, the first N column vectors of $T(u,\tau,\check{\eta})$, we ask that

$$\frac{\partial}{\partial \tau} \det\left(B(0)\, R_1^-(0,\tau,\check{\eta}),\dots,B(0)\, R_N^-(0,\tau,\check{\eta})\right)\Big|_{(\tau,\check{\eta})=\underline{\eta}} \neq 0\,.$$

In this framework, our main result reads as follows.

Theorem 6. *Let Assumptions 1, 2, 3 and 4 be satisfied. Then the equation governing weakly nonlinear surface wave solutions to* (99)*, namely* (103) *below, satisfies the evolutionarity condition $\alpha_0 \neq 0$. Furthermore, if we rewrite this equation under the abstract form* (45)-(46)*, then Hunter's stability condition $q(1,0^+) = q(-1,0^+)$ is satisfied. In particular, the corresponding Cauchy problem is locally well-posed in Sobolev spaces $H^m(\mathbb{R})$, $m \geq 2$.*

Let us make a few last comments before giving the proof of Theorem 6. For hyperbolic boundary value problems that do not satisfy the uniform Lopatinskii condition, the most favorable situation is the case when the uniform Lopatinskii condition fails in the elliptic region, which is the case considered in [22] (we also refer to [8, chapter 7] for a detailed treatment of linear problems with constant coefficients). However, according to the classification pointed out in [7], the degeneracy of the uniform Lopatinskii condition in the elliptic region is unstable with respect to perturbations of either the hyperbolic operator or the boundary conditions. In other words, if we go back to Assumption 3, the verification of the Lopatinskii condition for the linearized problem (100) at $u = 0$ does not necessarily imply that the linearized problem (100) at a different $u \in \mathscr{M}$ (close to 0) also satisfies the Lopatinskii condition (and of course we cannot say anything about the Lopatinskii condition at $u \notin \mathscr{M}$). An example given at the end of this Section illustrates this fact. Consequently, the assumptions made in Theorem 6 represent more or less the minimal price to pay in order to get energy estimates without loss of derivatives and local solvability for (99) in the framework of Sobolev spaces (of sufficiently high order)[9]. Our main result states that in this situation, the equation governing weakly nonlinear surface waves is also locally well-posed in Sobolev spaces (of sufficiently high order). In particular, if one follows the analysis of [15], Theorem 6 confirms the validity of weakly nonlinear geometric optics expansions for (99) provided that we reinforce a little bit Assumption 3 in order to make sure that (99) is indeed locally well-posed. Even though this result may not seem very surprising (it was more or less conjectured by Hunter, see [12, page 193]), it is in sharp contrast with the case of hyperbolic boundary value problems for which the uniform Lopatinskii condition fails in the hyperbolic region, in which there is a loss of derivatives. Indeed, the results in [10] point out an amplification of oscillations at the boundary in that situation, which means that the regime of weakly nonlinear geometric optics corresponds to initial oscillations with a much smaller amplitude than what we consider here.

3.2 The amplitude equation for weakly nonlinear surface waves: a reminder

In this paragraph, we introduce a few notations and recall the main steps in the derivation of the amplitude equation. We feel free to skip some computations because they involve very few modifications compared to the original work by Hunter [12]. (The main differences are that here we do not require that the hyperbolic system be in conservative form, and we consider nonlinear boundary conditions to allow for more general situations, but the impact on the computations is harmless.)

We let $\underline{R}_n^-$, $n = 1, \dots, N$, and $\underline{R}_n^+$, $n = 1, \dots, N$, denote the first N, and last N column vectors of the matrix $T(0, \underline{\eta})$. From now on, quantities that depend on u and/or $(\tau, \check{\eta})$ in a neighborhood of $(0, \underline{\eta})$ are denoted with an underline when evaluated at $(0, \underline{\eta})$. Since $-i\, A_d(0)^{-1}\, \underline{A}$ has purely imaginary coefficients, its eigenmodes are pairwise, and there is no loss of generality in assuming that

$$\forall\, n = 1, \dots, N\,, \quad \underline{\beta}_n^+ = -\overline{\underline{\beta}_n^-}\,, \quad \underline{R}_n^+ = \overline{\underline{R}_n^-}\,.$$

The vectors $\underline{R}_n^-$ span the stable subspace of $-i\, A_d(0)^{-1}\, \underline{A}$, that is $\underline{\mathbb{E}}^s$, see Theorem 5. (Recall that at elliptic frequencies, the continuous extension of the stable subspace coincides with the 'true' stable subspace, whereas at non-elliptic frequencies, the continuous

[9] To obtain energy estimates and solvability, one also needs to study the Lopatinskii condition in the so-called hyperbolic region and at glancing frequencies but this has no impact on our analysis.

extension of the stable subspace also contains part of the central subspace.) Moreover, Assumption 4 shows that $\underline{R}_n^{\pm}$ is an eigenvector of $-i\,A_d(0)^{-1}\,\underline{A}$ associated with the eigenvalue $\underline{\beta}_n^{\pm}$. According to Assumption 3, we know that the intersection $\underline{\mathbb{E}}^s \cap \ker B(0)$ is not trivial, and Assumption 4 implies (by Lemma A.5) that the latter vector space is (exactly) one-dimensional. Thus there exist $\gamma_1, \dots, \gamma_N \in \mathbb{C}$ such that

$$\underline{\mathbb{E}}^s \cap \ker B(0) = \text{Span}\, V\,, \quad V := \sum_{n=1}^{N} \gamma_n\, \underline{R}_n^- \neq 0\,.$$

Following [12], we also consider some vectors $\underline{L}_n^{\pm} \in \mathbb{C}^{2N}$ that satisfy

$$\forall\, n = 1, \dots, N\,, \quad (\underline{L}_n^{\pm})^*(i\,\underline{A} + \underline{\beta}_n^{\pm}\, A_d(0)) = 0\,, \quad \underline{L}_n^{+} = \overline{\underline{L}_n^{-}}\,.$$

The vectors $\underline{L}_n^{\pm}$ are normalized with the convention

$$(\underline{L}_n^{\mp})^*\, A_d(0)\, \underline{R}_m^{\pm} = 0\,, \quad (\underline{L}_n^{\pm})^*\, A_d(0)\, \underline{R}_m^{\pm} = \begin{cases} 1\,, & \text{if } n = m, \\ 0\,, & \text{otherwise.} \end{cases}$$

Eventually, we consider a nonzero vector $\sigma \in \mathbb{C}^N$ satisfying

$$\forall\, n = 1, \dots, N\,, \quad \sigma^*\, B(0)\, \underline{R}_n^- = 0\,,$$

or equivalently

$$\forall\, X \in \underline{\mathbb{E}}^s\,, \quad \sigma^*\, B(0)\, X = 0\,.$$

Let us observe that σ is uniquely defined up to a multiplicative constant because the vector space $B(0)\,\underline{\mathbb{E}}^s$ is a hyperplane of $\mathbb{C}^N$, see the discussion above. A convenient choice of σ for the analysis will be made below but, of course, the amplitude equation (103) for weakly nonlinear surface waves derived below does not depend on such a choice.

We formally look for asymptotic solutions to (99) of the form

$$u_\varepsilon(t,x) = \varepsilon\, u_1(\varepsilon\, t, \underline{\eta}_0\, t + \check{\underline{\eta}} \cdot \check{x}, x_d) + \varepsilon^2\, u_2(\varepsilon\, t, \underline{\eta}_0\, t + \check{\underline{\eta}} \cdot \check{x}, x_d) + O(\varepsilon^3)\,,$$

with $u_1(s,y,+\infty) = u_2(s,y,+\infty) = 0$. As in Section 2, we derive boundary value problems for u_1 and u_2 by equating to zero the $O(\varepsilon)$ and $O(\varepsilon^2)$ terms. See [12, section 2] or [6] for detailed computations. In particular, u_1 should solve

$$\begin{cases} \underline{A}\, \partial_y u_1 + A_d(0)\, \partial_z u_1 = 0\,, & y \in \mathbb{R}\,, z > 0\,, \\ B(0)\, u_1|_{z=0} = 0\,, \end{cases}$$

so that we can write

$$u_1(s,y,z) = (v(s,\cdot) * r(\cdot, z))(y)\,,$$

where $v(s,y) \in \mathbb{R}$, and the y-Fourier transform $\widehat{r}$ of r is defined by

$$\widehat{r}(k,z) := \begin{cases} \sum_{n=1}^{N} \gamma_n\, \mathrm{e}^{\,\underline{\beta}_n^-\, k\, z}\, \underline{R}_n^-\,, & \text{if } k > 0, \\ \sum_{n=1}^{N} \overline{\gamma_n}\, \mathrm{e}^{\,\underline{\beta}_n^+\, k\, z}\, \underline{R}_n^+\,, & \text{if } k < 0. \end{cases}$$

The boundary value problem satisfied by u_2 takes the form

$$\begin{cases} \underline{A}\,\partial_y u_2 + A_d(0)\,\partial_z u_2 + F_1 = 0\,, & y \in \mathbb{R}\,, z > 0\,,\\ B(0)\,u_2|_{z=0} + g_1 = 0\,, \end{cases}$$

with

$$\begin{aligned} F_1 &:= A_0(0)\,\partial_s u_1 + (\mathrm{d}A(0,\underline{\eta})\cdot u_1)\,\partial_y u_1 + (\mathrm{d}A_d(0)\cdot u_1)\,\partial_z u_1\,,\\ g_1 &:= (\mathrm{d}B(0)\cdot u_1)\,u_1|_{z=0}\,. \end{aligned}$$

By the same argument as in [12] (see also [6], or Proposition 5 above), we find that a necessary condition for the existence of a solution u_2 to the latter boundary value problem with $u_2(s,y,+\infty)=0$ is

$$\forall\, k \neq 0\,, \quad \int_0^{+\infty} L(k,z)\,\widehat{F_1}(s,k,z)\,\mathrm{d}z + \sigma(k)\,\widehat{g_1}(s,k) = 0\,, \tag{102}$$

where we have used the notation

$$L(k,z) := \begin{cases} \sum_{n=1}^N \sigma^*\,B(0)\,\underline{R}_n^+\,\mathrm{e}^{-\underline{\beta}_n^+\,k\,z}\,(\underline{L}_n^+)^*\,, & \text{if } k>0,\\ \sum_{n=1}^N \overline{\sigma}^*\,B(0)\,\underline{R}_n^-\,\mathrm{e}^{-\underline{\beta}_n^-\,k\,z}\,(\underline{L}_n^-)^*\,, & \text{if } k<0, \end{cases} \qquad \sigma(k) := \begin{cases} \sigma^*\,, & \text{if } k>0,\\ \overline{\sigma}^*\,, & \text{if } k<0. \end{cases}$$

Equation (102) can be rewritten for the unknown scalar function $w := \mathscr{F}_y v$ by using the above definitions of F_1 and g_1, and the classical formula $2\,\pi\,\mathscr{F}(f\times g) = \mathscr{F}(f) * \mathscr{F}(g)$. We obtain

$$a_0(k)\,\partial_s w + \int_{\mathbb{R}} a_1(k-k',k')\,w(s,k-k')\,w(s,k')\,\mathrm{d}k' = 0\,, \tag{103}$$

with

$$a_0(k) := \begin{cases} \alpha_0/k\,, & \text{if } k>0,\\ -\overline{\alpha_0}/k\,, & \text{if } k<0, \end{cases} \qquad \alpha_0 := \sum_{p,q=1}^N \frac{\sigma^*\,B(0)\,\underline{R}_p^+}{\underline{\beta}_p^+ - \underline{\beta}_q^-}\,(\underline{L}_p^+)^*\,A_0(0)\,(\gamma_q\,\underline{R}_q^-)\,, \tag{104}$$

and

$$\begin{aligned} a_1(k,k') :=& \frac{i\,k'}{2\,\pi}\int_0^{+\infty} L(k+k',z)\,(\mathrm{d}A(0,\underline{\eta})\cdot\widehat{r}(k,z))\,\widehat{r}(k',z)\,\mathrm{d}z\\ &+ \frac{1}{2\,\pi}\int_0^{+\infty} L(k+k',z)\,(\mathrm{d}A_d(0)\cdot\widehat{r}(k,z))\,\partial_z\widehat{r}(k',z)\,\mathrm{d}z\\ &+ \frac{1}{2\,\pi}\,\sigma(k+k')\,(\mathrm{d}B(0)\cdot\widehat{r}(k,0))\,\widehat{r}(k',0)\,. \end{aligned} \tag{105}$$

The definition of α_0 is legitimate because $\underline{\beta}_p^+ - \underline{\beta}_q^-$ has positive real part so it cannot be zero.

If we can prove that α_0 is nonzero - and this does not seem obvious by looking at the formula (104) - then (103) can be recast under the form (45), (46), with

$$q(k,k') := \frac{a_1(k,k') + a_1(k',k)}{2\,i\,(k+k')\,a_0(k+k')}\,, \tag{106}$$

so Hunter's stability condition (49) amounts to

$$\frac{a_1(1,0^+) + a_1(0^+,1)}{i\,\alpha_0} = \frac{a_1(-1,0^+) + a_1(0^+,-1)}{\overline{i\,\alpha_0}}\,.$$

The quantities $a_1(1,0^+)$, $a_1(0^+,1)$ etc. can be computed by using the definition (105) of the kernel a_1 and the above expressions for $\widehat{r}$, L and σ. In particular, Hunter's stability condition can be characterized in the following way (the argument is entirely similar to [12, page 193] and [6, Proposition 2.6]).

Lemma 3. *In the framework of Theorem 6, if α_0 defined by* (104) *is nonzero, then the kernel q in* (106) *satisfies Hunter's stability condition $q(1,0^+) = q(-1,0^+)$ if and only if the linear form*

$$a\,:\,X \in \mathbb{R}^{2N} \longmapsto \frac{1}{\alpha_0}\,\frac{\sigma^*\,B(0)\,\underline{R}_p^+}{\underline{\beta}_p^+ - \underline{\beta}_q^-}\,(\underline{L}_p^+)^*\,\big(\mathrm{d}_u A(0,\underline{\eta})\cdot X - i\,\underline{\beta}_q^-\,\mathrm{d}_u A_d(0)\cdot X\big)\,(\gamma_q\,\underline{R}_q^-)$$
$$- \frac{i\,\sigma^*}{\alpha_0}\,\big((\mathrm{d}B(0)\cdot X)\,V + (\mathrm{d}B(0)\cdot V)\,X\big)\,,$$

takes real values on the vectors $P := \mathsf{Re}\,V$ and $Q := \mathsf{Im}\,V$. (We recall that V spans $\underline{\mathbb{E}}^s \cap \ker B(0)$.)

3.3 Evolutionarity of the amplitude equation

In view of Theorem 3 in Section 2, the reader will not be surprised that the constant α_0 defined in (104) is proportional to the derivative of the Lopatinskii determinant with respect to τ.

Using Assumption 4, we can construct a basis $E_1(u,\tau,\check{\eta}),\dots,E_N(u,\tau,\check{\eta})$ of $\mathbb{E}^s(u,\tau,\check{\eta})$ that depends in a smooth (meaning $\mathscr{C}^1$) way on $(u,\tau,\check{\eta})$ and such that

$$\underline{E}_1 = \sum_{n=1}^N \gamma_n\,\underline{R}_n^-\,,$$

and the associated Lopatinskii determinant

$$\Delta(u,\tau,\check{\eta}) := \det_{n=1,\dots,N}\Big((\mathrm{d}B(u)\cdot E_n(u,\tau,\check{\eta}))\,u + B(u)\,E_n(u,\tau,\check{\eta})\Big)\,, \tag{107}$$

satisfies

$$\underline{\Delta} = 0\,,\quad \partial_\tau \Delta(0,\underline{\eta}) \neq 0\,.$$

We recall that the underline stands for evaluation at $u = 0$ and $(\tau,\check{\eta}) = \underline{\eta}$. Since the first column vector $B(0)\,\underline{E}_1$ in the determinant $\underline{\Delta}$ vanishes, we first obtain

$$\partial_\tau \Delta(0,\underline{\eta}) = \det(B(0)\,\partial_\tau E_1(0,\underline{\eta}), B(0)\,\underline{E}_2,\dots,B(0)\,\underline{E}_N)\,.$$

At this stage, the analysis becomes very similar to the proof of Theorem 3, see also [10, Proposition 3.5]. The vectors $B(0)\,\underline{E}_2,\dots,B(0)\,\underline{E}_N$ form a basis of the hyperplane $B(0)\,\underline{\mathbb{E}}^s$. Consequently the two linear forms

$$X \in \mathbb{C}^N \longmapsto \det(X, B(0)\,\underline{E}_2,\dots,B(0)\,\underline{E}_N)\,,\quad \text{and } X \in \mathbb{C}^N \longmapsto \sigma^*\,X\,,$$

are proportional one to the other. So there is no loss of generality in choosing σ such that

$$\forall\, X \in \mathbb{C}^N\,, \quad \det(X, B(0)\,\underline{E}_2, \dots, B(0)\,\underline{E}_N) = \sigma^*\, X\,, \tag{108}$$

hence the simple relation

$$\partial_\tau \Delta(0,\underline{\eta}) = \sigma^*\, B(0)\, \partial_\tau E_1(\underline{\eta}) \neq 0\,. \tag{109}$$

Now, decomposing the vector $E_1(0,\tau,\check{\eta})$ along each eigenvector $R_q^-(0,\tau,\check{\eta})$ of the matrix $-i\, A_d(0)^{-1}\, A(0,\tau,\check{\eta})$, we have

$$E_1(0,\tau,\check{\eta}) \;=\; \sum_{q=1}^N E_{1,q}(0,\tau,\check{\eta})\,, \quad \big(i\, A(0,\tau,\check{\eta}) \,+\, \beta_q^-(0,\tau,\check{\eta})\, A_d(0)\big)\, E_{1,q}(0,\tau,\check{\eta}) \;=\; 0\,,$$

with $\underline{E}_{1,q} = \gamma_q\, \underline{R}_q^-$. We differentiate each equation satisfied by $E_{1,q}$ with respect to τ and evaluate at $\underline{\eta}$, thus obtaining

$$\big(i\, A_0(0) + \partial_\tau \beta_q^-(0,\underline{\eta})\, A_d(0)\big)\, \gamma_q\, \underline{R}_q^- + \big(i\, \underline{A} + \,\underline{\beta}_q^-\, A_d(0)\big)\, \partial_\tau E_{1,q}(0,\underline{\eta}) = 0\,.$$

Each vector $\partial_\tau E_{1,q}(0,\underline{\eta})$ can be decomposed on the basis formed by the vectors $\underline{R}_p^\pm$, $p = 1,\dots,N$, and the coordinate on $\underline{R}_p^+$ is obtained by multiplying the latter relation by $(\underline{L}_p^+)^*$. We obtain

$$\partial_\tau E_{1,q}(0,\underline{\eta}) - i \sum_{p=1}^N \frac{(\underline{L}_p^+)^*\, A_0(0)\, \gamma_q\, \underline{R}_q^-}{\underline{\beta}_p^+ - \,\underline{\beta}_q^-}\, \underline{R}_p^+ \in \underline{\mathbb{E}}^s\,.$$

Then (109) turns into

$$\partial_\tau \Delta(0,\underline{\eta}) = \sum_{q=1}^N \sigma^*\, B(0)\, \partial_\tau E_{1,q}(0,\underline{\eta}) = i \sum_{p,q=1}^N \frac{\sigma^*\, B(0)\, \underline{R}_p^+}{\underline{\beta}_p^+ - \,\underline{\beta}_q^-}\, (\underline{L}_p^+)^*\, A_0(0)\, \gamma_q\, \underline{R}_q^- = i\, \alpha_0\,, \tag{110}$$

where we have used that $\sigma^*\, B(0)\, \underline{R}_p^- = 0$. As claimed in Theorem 6, we have shown that α_0 is nonzero.

3.4 Verification of Hunter's stability condition

In the preceeding paragraph, we have fixed the state u for (100) at 0, and we have differentiated the corresponding Lopatinskii determinant with respect to τ. In this paragraph, we are going to fix $(\tau,\check{\eta}) = \underline{\eta}$, and differentiate the Lopatinskii determinant $\Delta(u,\underline{\eta})$ with respect to u. We recall the definition (107) for Δ, and we also recall that in this formula, the first column vector of the determinant vanishes at $u = 0$ and $(\tau,\check{\eta}) = \underline{\eta}$. We thus obtain

$$\begin{aligned}&\forall\, X \in \mathbb{R}^{2N}\,, \quad \mathrm{d}_u \Delta(0,\underline{\eta}) \cdot X \\ &\qquad = \det\Big(\mathrm{d}_u [(\mathrm{d}B(u) \cdot E_n(u,\underline{\eta}))\, u + B(u)\, E_n(u,\underline{\eta})]\big|_{u=0} \cdot X, B(0)\, \underline{E}_2, \dots, B(0)\, \underline{E}_N\Big)\,.\end{aligned}$$

Computing the differential with respect to u, and recalling the choice (108) for σ, we obtain

$$\mathrm{d}_u\Delta(0,\underline{\eta})\cdot X=\sigma^*\left(\mathrm{d}B(0)\cdot V\right)X+\sigma^*\left(\mathrm{d}B(0)\cdot X\right)V+\sigma^*\,B(0)\,\mathrm{d}_uE_1(0,\underline{\eta})\cdot X\,. \quad (111)$$

Our goal now is to compute the derivative $\mathrm{d}_uE_1(0,\underline{\eta})\cdot X$ in (111) by applying the same technique as in the preceeding paragraph. Hence we introduce the decomposition

$$E_1(u,\underline{\eta})=\sum_{q=1}^{N}E_{1,q}(u,\underline{\eta})\,,\quad \big(i\,A(u,\underline{\eta})+\beta_q^-(u,\underline{\eta})\,A_d(u)\big)\,E_{1,q}(u,\underline{\eta})=0\,,\quad \underline{E}_{1,q}=\gamma_q\,\underline{R}_q^-\,.$$

We differentiate each equation satisfied by $E_{1,q}$ with respect to u in the direction X and evaluate at $u=0$, thus obtaining

$$\begin{aligned}\big(\mathrm{d}_uA(0,\underline{\eta})\cdot X-i\,\underline{\beta}_q^-\,\mathrm{d}A_d(0)\cdot X\big)\,\gamma_q\,\underline{R}_q^- - i\,(\mathrm{d}_u\beta_q^-(0,\underline{\eta})\cdot X)\,A_d(0)\,\gamma_q\,\underline{R}_q^-&\\ +\big(\underline{A}-i\,\underline{\beta}_q^-\,A_d(0)\big)\,\mathrm{d}_uE_{1,q}(0,\underline{\eta})\cdot X=0\,.&\end{aligned}$$

Each vector $\mathrm{d}_uE_{1,q}(0,\underline{\eta})\cdot X$ can be decomposed on the basis formed by the vectors $\underline{R}_p^\pm$, $p=1,\dots,N$, and the coordinate on $\underline{R}_p^+$ is again obtained by multiplying the latter relation by $(\underline{L}_p^+)^*$. We obtain

$$\mathrm{d}_uE_{1,q}(0,\underline{\eta})\cdot X-i\sum_{p=1}^{N}\frac{(\underline{L}_p^+)^*\big(\mathrm{d}_uA(0,\underline{\eta})\cdot X-i\,\underline{\beta}_q^-\,\mathrm{d}A_d(0)\cdot X\big)\,\gamma_q\,\underline{R}_q^-}{\underline{\beta}_p^+-\underline{\beta}_q^-}\,\underline{R}_p^+\in\underline{\mathbb{E}}^s\,.$$

We can then simplify (111) and get

$$\begin{aligned}\mathrm{d}_u\Delta(0,\underline{\eta})\cdot X&=\sigma^*\left(\mathrm{d}B(0)\cdot V\right)X+\sigma^*\left(\mathrm{d}B(0)\cdot X\right)V\\&\quad+i\sum_{p,q=1}^{N}\frac{\sigma^*\,B(0)\,\underline{R}_p^+}{\underline{\beta}_p^+-\underline{\beta}_q^-}\,(\underline{L}_p^+)^*\,\big(\mathrm{d}_uA(0,\underline{\eta})\cdot X-i\,\underline{\beta}_q^-\,\mathrm{d}A_d(0)\cdot X\big)\,\gamma_q\,\underline{R}_q^-\,. \quad (112)\end{aligned}$$

Comparing the expression (112) with the linear form a defined in Lemma 3, we have

$$\forall\,X\in\mathbb{R}^{2N}\,,\quad \mathrm{d}_u\Delta(0,\underline{\eta})\cdot X=i\,\alpha_0\,a(X)=\partial_\tau\Delta(0,\underline{\eta})\,a(X)\,, \quad (113)$$

where we have used (110).

The conclusion of the proof of Theorem 6 relies on the implicit function Theorem. More precisely, we know that there exists a $\mathscr{C}^1$ function Θ defined in a sufficiently small neighborhood of $(0,\underline{\check\eta})$ in $\mathscr{O}\times\mathbb{R}^{d-1}$ with values in a neighborhood of $\underline{\eta}_0$, such that

$$\Delta(u,\Theta(u,\check\eta),\check\eta)=0\,,\quad \Theta(0,\underline{\check\eta})=\underline{\eta}_0\,.$$

Differentiating with respect to u and recalling (113), we have $\mathrm{d}_u\Theta(0,\underline{\check\eta})=-a$. Let us now observe that Assumption 3 implies the following property: for all $u\in\mathscr{M}$, $\Delta(u,\tau,\check\eta)$ is nonzero for τ of negative imaginary part (this is an equivalent formulation of the Lopatinskii condition). In particular, $\Theta(u,\check\eta)$ has nonpositive imaginary part for all $u\in\mathscr{M}$ sufficiently close to 0. Differentiating with respect to $u\in\mathscr{M}$, we get

$$\forall\,X\in T_0\mathscr{M}\,,\quad a(X)=-\mathrm{d}_u\Theta(0,\underline{\check\eta})\cdot X\in\mathbb{R}\,.$$

By Lemma 2, we know that the tangent space $T_0\mathscr{M}$ coincides with $\ker B(0)$, and since $V\in\ker B(0)$, we also have $P,Q\in\ker B(0)$, so $a(P),a(Q)\in\mathbb{R}$. Lemma 3 thus implies that Hunter's stability condition (49) is satisfied.

3.5 An example

In this paragraph, we show on an example that Theorem 6 only gives sufficient conditions for the well-posedness of the amplitude equation (103) governing weakly nonlinear surface waves. More precisely, it is possible to perform formal asymptotic expansions for weakly nonlinear surface waves, and to derive a well-posed amplitude equation even though the original hyperbolic initial boundary value problem does not satisfy Assumption 3 (which was our main stability assumption for the original problem (99)). Of course, the rigorous justification of geometric optics expansion in this case may be completely out of reach by the usual techniques.

Let us consider the following symmetric hyperbolic initial boundary value problem

$$\begin{cases} \partial_t U + A_1\,\partial_1 U + A_2\,\partial_2 U = 0\,, & x \in \Omega\,, t > 0\,, \\ B(U)\,U = 0\,, & x \in \partial\Omega\,, t > 0\,, \\ U_{|t=0} = U_0\,, & x \in \Omega\,, \end{cases} \tag{114}$$

where the unknown U belongs to $\mathbb{R}^4$, the (constant) real symmetric matrices A_1, A_2 are defined by

$$A_1 := \begin{pmatrix} \mathbf{0}_2 & I_2 \\ I_2 & \mathbf{0}_2 \end{pmatrix}\,, \quad A_2 := \begin{pmatrix} I_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & -I_2 \end{pmatrix}\,,$$

and where the nonlinearity arises only in the boundary conditions. More precisely, we consider the matrix

$$B(U) := \begin{pmatrix} 1 & 0 & -U_3^2 & 1 \\ 0 & 1 & -1 & -U_3^2 \end{pmatrix}\,,$$

for the boundary conditions in (114).

It is a simple exercise to check that Assumption 1 is satisfied for the hyperbolic operator in (114). There are two distinct eigenvalues, each with multiplicity 2, and Assumption 2 is also trivially satisfied. The linearized problem at 0 has maximally dissipative boundary conditions, so the Lopatinskii condition is satisfied. Moreover, the corresponding stable subspace $\mathbb{E}^s(\tau, \check{\eta})$, $\mathsf{Im}\,\tau \leq 0$, $\check{\eta} \in \mathbb{R}$, is spanned by the vectors

$$\begin{pmatrix} \tau - \omega \\ 0 \\ -\check{\eta} \\ 0 \end{pmatrix}\,, \quad \begin{pmatrix} 0 \\ \tau - \omega \\ 0 \\ -\check{\eta} \end{pmatrix} \in \mathbb{C}^4\,,$$

where $\omega^2 = \tau^2 - \check{\eta}^2$, and ω has positive imaginary part when τ has negative imaginary part. With this parametrization of the stable subspace, the Lopatinskii determinant defined in (107) reads

$$\Delta(U, \tau, \check{\eta}) = \left(\tau - \omega + 3\,U_3^2\,\check{\eta}\right)\left(\tau - \omega + U_3^2\,\check{\eta}\right) + (1 + 2\,U_3\,U_4)\,\check{\eta}^2\,. \tag{115}$$

From the expression $\Delta(0, \tau, \check{\eta}) = 2\,\tau\,(\tau - \omega)$, we find that the linearized problem at 0 admits surface waves of the form $\exp(i\,\check{\eta}\,x_1 - |\check{\eta}|\,x_2)\,U$, for some $U \in \mathbb{E}^s(0, \check{\eta})$. Since 0 is a simple root of $\Delta(0, \cdot, \check{\eta})$, we can also check that Assumption 4 is satisfied. This implies that the above derivation of the amplitude equation for weakly nonlinear surface waves can be carried out. Observing that $\mathrm{d}B$ vanishes at zero, and recalling that A_1 and A_2 are constant, the amplitude equation (103) reads $\partial_s w = 0$, which is obviously well-posed. However, we are going to show the existence of a sequence of points converging to 0 in the manifold $\mathscr{M}$, around which the linearization of (114) does not satisfy the Lopatinskii condition.

In our example (114), the manifold $\mathscr{M}$ reads $\{U \in \mathbb{R}^4 \,/\, U_1 = U_3^3 - U_4\,,\, U_2 = U_3 + U_3^2\, U_4\}$. Let us consider a sequence of points in $\mathscr{M}$ with $U_4 = 0$, and $U_3 = \sqrt{\varepsilon_n}$ where (ε_n) is any sequence of positive numbers that converges to zero. We compute the Lopatinskii determinant for the linearization of (114) around the corresponding state $U^{(n)} \in \mathscr{M}$ by using the expression (115):

$$\Delta(U^{(n)}, \tau, \check{\eta}) = 2\,\tau\,(\tau - \omega) + 4\,\varepsilon_n\,\check{\eta}\,(\tau - \omega) + 3\,\varepsilon_n^2\,\check{\eta}^2\,, \quad \omega^2 = \tau^2 - \check{\eta}^2\,.$$

The Weierstrass preparation theorem shows that for n sufficiently large (or ε_n sufficiently small), $\Delta(U^{(n)}, \cdot, 1)$ has a unique root τ_n close to 0, and

$$\tau_n = -2\,\varepsilon_n - \frac{3\,i}{2}\,\varepsilon_n^2 + o(\varepsilon_n^2)\,.$$

In particular, τ_n has negative imaginary part for n sufficiently large, so the linearized problem around $U^{(n)}$ does not satisfy the Lopatinskii condition. Consequently, the original problem (114) does not satisfy Assumption 3, even though the amplitude equation for weakly nonlinear surface waves satisfies Hunter's stability condition.

The above example illustrates the fact that hyperbolic initial boundary value problems for which the uniform Lopatinskii condition is violated in the elliptic region can be unstable with respect to perturbations of the coefficients (this was already shown in [7] by abstract arguments).

Appendix

Scale invariance. Let us rephrase Lemma 1 in more algebraic terms, for which we refer to Olver's textbook [18]. The *evolutionary representative* of the (generalized) vector field

$$\mathbf{v} = x_i \partial_{x_i} \,+\, t\,\partial_t \,+\, \theta_\alpha\, u_\alpha\, \partial_{u_\alpha}$$

is $\mathbf{v}_P = P_\alpha \partial_{u_\alpha}$ where

$$P_\alpha := (\theta u)_\alpha - x_i u_{\alpha,i} - t u_{\alpha,t}$$

is called the *characteristic* of $\mathbf{v}_P$. Here above, $(\theta u)_\alpha$ stands for $\theta_\alpha u_\alpha$ *without summation*, $u_{\alpha,i}$ stands for $\partial_i u_\alpha$, and $u_{\alpha,t}$ for $\partial_t u_\alpha$. This is a usual notation, which we use repeatedly below also for second and third order derivatives ($u_{\alpha,ij}$ stands for $\partial_i \partial_j u_\alpha$, etc.). We also consider the generalized vector fields $\mathbf{v}_Q$ and $\mathbf{v}_R$ whose characteristics are given by

$$Q_\alpha := J_{\alpha\gamma}\,(\underline{a}_{\gamma\beta}\, u_\beta \,+\, (\underline{b}_{\gamma\beta j} \,-\, \underline{b}_{\beta\gamma j})\, u_{\beta,j} \,-\, \underline{c}_{\gamma j \beta \ell}\, u_{\beta,j\ell})\,,$$

$$R_\alpha := \underline{b}_{\beta\alpha d}\, u_\beta \,+\, \underline{c}_{\alpha d \beta \ell}\, u_{\beta,\ell}\,.$$

Then we have the following.

Lemma A.1. *The system of evolution equations $\partial_t u_\alpha = Q_\alpha$ for $\alpha \in \{1,\dots,N\}$ is invariant by the symmetry group generated by $\mathbf{v}$ if and only if, for all $\alpha,\beta \in \{1,\dots,N\}$ (there is no summation convention over these indices in the equations below),*

$$(-1 + \theta_\alpha - \theta_\beta)\, J_{\alpha\gamma}\, \underline{a}_{\gamma\beta} = 0\,, \tag{A.1}$$

$$(\theta_\alpha - \theta_\beta)\, J_{\alpha\gamma}\, (\underline{b}_{\beta\gamma j} - \underline{b}_{\gamma\beta j}) = 0\,, \quad \forall j \in \{1,\dots,d\}\,, \tag{A.2}$$

$$(1 + \theta_\alpha - \theta_\beta)\, J_{\alpha\gamma}\, \underline{c}_{\gamma j\beta\ell} = 0\,, \quad \forall j,\ell \in \{1,\dots,d\}\,. \tag{A.3}$$

The set of equations $R_\alpha = 0$ for $\alpha \in \{1,\dots,N\}$ is invariant by the symmetry group generated by $\mathbf{v}$ if and only if there exist real numbers θ^α such that for all $\alpha,\beta \in \{1,\dots,N\}$ (with no summation convention over these indices below)

$$(\theta_\beta - \theta^\alpha)\, \underline{b}_{\beta\alpha d} = 0 \ \text{ and } (\theta_\beta - \theta^\alpha - 1)\, \underline{c}_{\beta\alpha d\ell} = 0\ ,\ \ \forall \ell \in \{1,\dots,d\}\,.$$

Proof. Invariance properties can be characterized by using the *prolongations* of $\mathbf{v}_P$ and $\mathbf{v}_Q$, defined as

$$\mathrm{pr}\mathbf{v}_P = P_\beta \partial_{u_\beta} + (D_j P_\beta)\, \partial_{u_{\beta,j}} + (D_t P_\beta)\, \partial_{u_{\beta,t}} + (D_{jk} P_\beta)\, \partial_{u_{\beta,jk}} + \dots$$

$$\mathrm{pr}\mathbf{v}_Q = Q_\beta \partial_{u_\beta} + (D_j Q_\beta)\, \partial_{u_{\beta,j}} + (D_t Q_\beta)\, \partial_{u_{\beta,t}} + \dots$$

The system of evolution equations $\partial_t u_\alpha = Q_\alpha$ for $\alpha \in \{1,\dots,N\}$ is invariant by the symmetry group generated by $\mathbf{v}$ if and only if (see [17, p.303])

$$\partial_t P_\alpha + \mathrm{pr}\mathbf{v}_Q(P_\alpha) - \mathrm{pr}\mathbf{v}_P(Q_\alpha) = 0\,, \quad \alpha \in \{1,\dots,N\}\,. \tag{A.4}$$

By plugging in this equation the following expressions of total derivatives

$$\begin{aligned}
D_t Q_\alpha &= J_{\alpha\gamma}\, (\underline{a}_{\gamma\beta}\, u_{\beta,t} + (\underline{b}_{\gamma\beta j} - \underline{b}_{\beta\gamma j})\, u_{\beta,jt} - \underline{c}_{\gamma j\beta\ell}\, u_{\beta,j\ell t})\,,\\
D_i Q_\alpha &= J_{\alpha\gamma}\, (\underline{a}_{\gamma\beta}\, u_{\beta,i} + (\underline{b}_{\gamma\beta j} - \underline{b}_{\beta\gamma j})\, u_{\beta,ji} - \underline{c}_{\gamma j\beta\ell}\, u_{\beta,j\ell i})\,,\\
D_j P_\beta &= (\theta u)_{\beta,j} - u_{\beta,j} - x_i u_{\beta,ij} - t u_{\beta,tj}\,,\\
D_{jk} P_\beta &= (\theta u)_{\beta,jk} - 2\, u_{\beta,jk} - x_i u_{\beta,ijk} - t u_{\beta,tjk}\,,
\end{aligned}$$

we arrive at

$$(-1+\theta_\alpha-\theta_\beta)\, J_{\alpha\gamma}\, \underline{a}_{\gamma\beta}\, u_\beta + (\theta_\alpha-\theta_\beta)\, J_{\alpha\gamma}\, (\underline{b}_{\beta\gamma j} - \underline{b}_{\gamma\beta j})\, u_{\beta,j} + (1+\theta_\alpha-\theta_\beta)\, J_{\alpha\gamma}\, \underline{c}_{\gamma j\beta k}\, u_{\beta,jk} = 0\,,$$

which must be satisfied whatever u, ∇u, and $\nabla^2 u$. In the equation above, there is no summation on α, and by taking u such that $u_\delta = 0$ for $\delta \neq \beta$, we also get the equation with no summation on β.

For the invariance of the boundary conditions $R_\alpha = 0$ for $\alpha \in \{1,\dots,N\}$ the iff condition is that $\mathrm{pr}\mathbf{v}_P(\underline{b}_{\beta\alpha d}\, u_\beta + \underline{c}_{\alpha d\beta\ell}\, u_{\beta,\ell}) = 0$ for all u such that $\underline{b}_{\beta\alpha d}\, u_\beta + \underline{c}_{\alpha d\beta\ell}\, u_{\beta,\ell} = 0$. Computing that

$$\mathrm{pr}\mathbf{v}_P(\underline{b}_{\beta\alpha d}\, u_\beta + \underline{c}_{\alpha d\beta\ell}\, u_{\beta,\ell}) =$$

$$\theta_\beta\, (\underline{b}_{\beta\alpha d}\, u_\beta + \underline{c}_{\alpha d\beta\ell}\, u_{\beta,\ell}) - x_i\, D_i(\underline{b}_{\beta\alpha d}\, u_\beta + \underline{c}_{\alpha d\beta\ell}\, u_{\beta,\ell}) - t\, D_t(\underline{b}_{\beta\alpha d}\, u_\beta + \underline{c}_{\alpha d\beta\ell}\, u_{\beta,\ell}) - u_{\beta,\ell}\, \underline{c}_{\alpha d\beta\ell}\,,$$

this leaves the condition

$$\forall u \in \mathbb{R}^n\,, \forall F \in \mathbb{R}^{N\times d}\,,\ (\underline{b}_{\beta\alpha d}\, u_\beta + \underline{c}_{\alpha d\beta\ell}\, F_{\beta\ell} = 0 \ \Rightarrow\ \theta_\beta\, \underline{b}_{\beta\alpha d}\, u_\beta + (\theta_\beta - 1)\, \underline{c}_{\alpha d\beta\ell}\, F_{\beta\ell} = 0)\,,$$

which is equivalent to the claimed property. □

Remark 8. *A reader not familiar with Lie group techniques may also check this result 'with bare hands'. This is easy since the vector fields* $\mathbf{v}$, $\mathbf{v}_Q$ *and* $\mathbf{v}_R$ *are linear, and* $\mathbf{v}$ *is even diagonal. The advantage of the Lie group approach is that it can also be used in a systematic way (even though it is very technical) to investigate the symmetries of the nonlinear problem* (4). *Indeed, the evolution equation* $\partial_t u = J\,\delta\mathscr{E}[u]$ *is invariant under* $\mathbf{v}$ *if and only if we have* (A.4) *with* Q_α *redefined as* $(J\,\delta\mathscr{E}[u])_\alpha$, *while the boundary condition* $(\delta_{\mathbf{n}}\mathscr{E}[u])_\alpha = 0$, *which equivalently reads* $f_\alpha(u,\nabla u) = 0$ *where* $f_\alpha := n_j \partial E/\partial F_{\alpha j}$, *is invariant if and only if*

$$f_\alpha = 0 \Rightarrow \mathrm{pr}\mathbf{v}_P(f_\alpha) = 0\,.$$

Isotropic linearized elasticity. We perform below in our notations the computations leading to the well-known Rayleigh waves. The equations of isotropic linearized elasticity are of the form (5) with a quadratic mapping $W : F \mapsto W(F)$. More precisely, let us consider

$$W(F) \;=\; \frac{\lambda}{2}\,(\mathrm{tr}F)^2 \;+\; \frac{\mu}{4}\sum_{\alpha,j}(F_{\alpha j}+F_{j\alpha})^2\,,$$

where λ and μ are real numbers called the Lamé coefficients of the material. We assume that at least μ is positive and $\mu+\lambda>0$. We have

$$\frac{\partial W}{\partial F_{\alpha j}} \;=\; \lambda\,(\mathrm{tr}F)\,\delta_{\alpha j} \;+\; \mu\,(F_{\alpha j}+F_{j\alpha})\,,$$

where $\delta_{\alpha j}$ denotes the Kronecker symbol (equal to one if $\alpha = j$, zero otherwise), hence with the notations introduced in Sections 2.1-2.2,

$$c_{\alpha j\beta\ell} \;=\; \lambda\delta_{\alpha j}\,\delta_{\beta\ell} \;+\; \mu\,(\delta_{\alpha\beta}\,\delta_{j\ell} \;+\; \delta_{\beta j}\,\delta_{\alpha\ell})\,.$$

In particular,

$$\begin{gathered}
c_{\alpha d\beta d} \;=\; \mu\,\delta_{\alpha\beta} \;+\; (\lambda+\mu)\delta_{\alpha d}\,\delta_{\beta d}\,,\\
c_{\alpha d\beta\ell} \;=\; \lambda\delta_{\alpha d}\,\delta_{\beta\ell} \;+\; \mu\,\delta_{\beta d}\,\delta_{\alpha\ell}\,, \quad \text{ if } \ell \le d-1\,,\\
c_{\alpha j\beta\ell} \;=\; 0\,, \quad \text{ if } (\alpha = d \text{ or } \beta = d, \alpha\neq\beta) \text{ and } j\le d-1\,,\, \ell\le d-1\,.\\
c_{djd\ell} \;=\; \mu\,\delta_{j\ell}\,, \quad \text{ if } j\le d-1\,,\, \ell\le d-1\,.
\end{gathered}$$

In matrix terms, this means

$$\check{\Lambda} \;=\; \left(\begin{array}{c|c} \mu I_{d-1} & 0 \\ \hline 0 & \lambda+2\mu \end{array}\right), \quad \check{A}^{\check{\eta}} \;=\; \left(\begin{array}{c|c} \mathbf{0}_{d-1} & \mu\,\check{\eta} \\ \hline \lambda\,\check{\eta}^T & 0 \end{array}\right), \quad \check{\Sigma}^{\check{\eta}} \;=\; \left(\begin{array}{c|c} \begin{array}{c}\mu\,\|\check{\eta}\|^2\,I_{d-1}\\ +(\lambda+\mu)\,\check{\eta}\otimes\check{\eta}\end{array} & 0 \\ \hline 0 & \mu\,\|\check{\eta}\|^2 \end{array}\right),$$

so that the BVP

$$\begin{aligned}
&\eta_0^2\,\widehat{\rho} \;=\; (\check{\Sigma}^{\check{\eta}} \;-\; i\,(\check{A}^{\check{\eta}} + (\check{A}^{\check{\eta}})^T)\,\partial_z \;-\; \check{\Lambda}\,\partial_z^2)\,\widehat{\rho} \;\text{ for }\; z>0\,,\\
&(i\,\check{A}^{\check{\eta}} \;+\; \check{\Lambda}\,\partial_z)\,\widehat{\rho} \;=\; 0 \;\text{ at }\; z=0\,,
\end{aligned}$$

is equivalent to

$$\widehat{\rho} = \begin{pmatrix} \widehat{\zeta} \\ \widehat{\sigma} \end{pmatrix}, \ \widehat{\zeta}(\eta, z) \in \mathbb{C}^{d-1}, \ \widehat{\sigma}(\eta, z) \in \mathbb{C},$$

and

$$\begin{cases} \eta_0^2 \,\widehat{\zeta} \,=\, \mu\|\check{\eta}\|^2\widehat{\zeta} \,+\, (\lambda+\mu)\,(\check{\eta}\cdot\widehat{\zeta})\,\check{\eta} \,-\, i\,(\lambda+\mu)\,\partial_z\widehat{\sigma} \,-\, \mu\,\partial_z^2\widehat{\zeta} \\ \eta_0^2\,\widehat{\sigma} \,=\, \mu\|\check{\eta}\|^2\widehat{\sigma} \,-\, i\,(\lambda+\mu)\,(\check{\eta}\cdot\partial_z\widehat{\zeta}) \,-\, (\lambda+2\mu)\,\partial_z^2\widehat{\sigma} \end{cases} \quad \text{for } z>0, \tag{A.5}$$

$$\begin{cases} i\,\mu\,\widehat{\sigma}\,\check{\eta} \,+\, \mu\partial_z\widehat{\zeta} \,=\, 0 \\ i\,\lambda\,(\check{\eta}\cdot\widehat{\zeta}) \,+\, (\lambda+2\mu)\,\partial_z\widehat{\sigma} \,=\, 0 \end{cases} \quad \text{at } z=0. \tag{A.6}$$

In particular, solutions of (A.5) of the form

$$\widehat{\rho} \,=\, \mathrm{e}^{-\omega z} \begin{pmatrix} \zeta \\ \sigma \end{pmatrix}, \ \zeta \in \mathbb{C}^{d-1}, \ \sigma \in \mathbb{C},$$

are found to be characterized by

$$\Big(\omega^2 \,=\, \|\check{\eta}\|^2 - \frac{\eta_0^2}{\mu} \ \text{ and } \check{\eta}\cdot\zeta + i\,\omega\,\sigma \,=\, 0\Big) \ \text{ or } \ \Big(\omega^2 \,=\, \|\check{\eta}\|^2 - \frac{\eta_0^2}{\lambda+2\mu} \ \text{ and } \begin{pmatrix} \zeta \\ \sigma \end{pmatrix} \parallel \begin{pmatrix} \check{\eta} \\ i\,\omega \end{pmatrix}\Big).$$

(To obtain this alternative we observe that if $\|\check{\eta}\|^2 \neq \eta_0^2/\mu$ then ζ must be parallel to $\check{\eta}$, and we are left with a 2×2 system in $(\check{\eta}\cdot\zeta, \sigma)$.) Furthermore, assuming that $\|\check{\eta}\|^2 > \eta_0^2/\mu$ and $\|\check{\eta}\|^2 > \eta_0^2/(\lambda+2\mu)$, and defining

$$\omega_1 \,:=\, \sqrt{\|\check{\eta}\|^2 \,-\, \frac{\eta_0^2}{\mu}}, \quad \omega_2 \,:=\, \sqrt{\|\check{\eta}\|^2 \,-\, \frac{\eta_0^2}{\lambda+2\mu}},$$

we see that solutions of (A.5)-(A.6) of the form

$$\widehat{\rho} \,=\, \mathrm{e}^{-\omega_1 z} \begin{pmatrix} \zeta_1 \\ \sigma_1 \end{pmatrix} + \mathrm{e}^{-\omega_2 z} \begin{pmatrix} \zeta_2 \\ \sigma_2 \end{pmatrix}, \ \begin{pmatrix} \zeta_1 \\ \sigma_1 \end{pmatrix} \perp \begin{pmatrix} \check{\eta} \\ i\,\omega_1 \end{pmatrix}, \ \begin{pmatrix} \zeta_2 \\ \sigma_2 \end{pmatrix} \parallel \begin{pmatrix} \check{\eta} \\ i\,\omega_2 \end{pmatrix},$$

are characterized by

$$\begin{cases} i\,\mu\,(\sigma_1 \,+\, \sigma_2)\,\check{\eta} \,-\, \mu(\omega_1\zeta_1 + \omega_2\zeta_2) \,=\, 0, \\ i\,\lambda\,\check{\eta}\cdot(\zeta_1 \,+\, \zeta_2) \,-\, (\lambda+2\mu)\,(\omega_1\sigma_1 + \omega_2\sigma_2) \,=\, 0. \end{cases}$$

Since $\mu\omega_1 > 0$ (by assumption), the first equation shows that ζ_1 must also be parallel to $\check{\eta}$, and using that $\check{\eta}\cdot\zeta_1 \,=\, -\,i\,\omega_1\,\sigma_1 \,=\, 0$, we are left with a 2×2 system in (σ_1, ν_2) where

$$\begin{pmatrix} \zeta_2 \\ \sigma_2 \end{pmatrix} \,=\, \nu_2 \begin{pmatrix} \check{\eta} \\ i\,\omega_2 \end{pmatrix},$$

namely

$$\begin{cases} i\,(\omega_1^2 \,+\, \|\check{\eta}\|^2)\,\sigma_1 \,-\, 2\,\omega_2\,\|\check{\eta}\|^2\,\nu_2 \,=\, 0, \\ 2\,\mu\,\omega_1\,\sigma_1 \,+\, i\,((\lambda+2\mu)\,\omega_2^2 \,-\, \lambda\,\|\check{\eta}\|^2)\,\nu_2 \,=\, 0. \end{cases}$$

Replacing $\omega_{1,2}^2$ by their expressions in terms of $\|\check{\eta}\|^2$, λ, and μ, we find that the existence of a nontrival solution of the latter system is equivalent to

$$\Big(\|\check{\eta}\|^2 \,-\, \frac{\eta_0^2}{2\mu}\Big)^2 \,-\, \omega_1\,\omega_2\,\|\check{\eta}\|^2 \,=\, 0\,, \tag{A.7}$$

which gives the equation in (80). Another claim of §2.4 is that we can choose σ_1 real: this is clear from the above system, choosing for instance $\nu_2 \,=\, i\,(\omega_1^2 + \|\check{\eta}\|^2)$ and $\sigma_1 \,=\, 2\,\omega_2\,\|\check{\eta}\|^2$.

Remark 9. *The actual existence of* Rayleigh waves *relies on the existence of real roots* η_0 *of* (A.7). *This is where we need* $\mu+\lambda>0$, *which ensures that* $c_S := \sqrt{\mu}$ *is less than* $c_P := \sqrt{2\mu+\lambda}$. *The former is the speed of* shear waves, *and the latter is the speed of* pressure waves. *If* $c_S < c_P$ *then the speed of Rayleigh waves,* $c_R := \underline{\eta}_0/\underline{\check{\eta}}$, *is given implicitly by*

$$(c_R^2/(2c_S^2)\,-\,1)^4 = \,(1-c_R^2/c_P^2)\,(1-c_R^2/c_S^2)$$

(which is just (A.7) *with different notations), and shown to be less than both* c_S *and* c_P.

Linear algebra.

Lemma A.2. *Let* J *be a skew-symmetric, real matrix, and* L *a symmetric, real matrix. If* L *is nonnegative then* $J\,L$ *has no eigenvalue of positive real part.*

Proof. This is a simplified version of Theorem 3.1 in [21], which holds true in much more generality (even in infinite dimensions). We first observe that if τ were an eigenvalue of positive real part of $J\,L$ then an associated eigenvector w_0 would be such that $w_0^*\,L\,w_0 = 0$. This can easily be shown by an ODE argument. Indeed, we observe that the solution of the Cauchy problem for $w' \,=\, J\,L\,w$ with initial data $w(0) \,=\, w_0$ is $w(t) \,=\, \mathrm{e}^{\tau t}\,w_0$, and such that $w(t)^*\,L\,w(t)$ is constant. Therefore, $\mathrm{e}^{(\tau+\overline{\tau})t}\,w_0^*\,L\,w_0 \,=\, w_0^*\,L\,w_0$ for all $t\in\mathbb{R}$, which implies $w_0^*\,L\,w_0 = 0$ since $\tau+\overline{\tau}\neq 0$. Now, there is an orthogonal decomposition $\ker L \oplus V = \mathbb{R}^N$, with $L_{|V}$ being positive. We thus infer that w_0 must belong to $\ker L$, which contradicts the fact that $J\,L\,w_0 = \tau\,w_0 \neq 0$. □

Lemma A.3. *Let* J *be a nonsingular, skew-adjoint* $N\times N$ *matrix such that* $J^2 = -I_N$, *and* H *a self-adjoint* $N\times N$ *matrix. Assume that the families of (complex) vectors* $(R_\alpha)_{\alpha\in\{1,\ldots,N\}}$ *and* $(L_\alpha)_{\alpha\in\{1,\ldots,N\}}$ *are such that*

$$J\,H\,R_\alpha \,=\, -\,\omega_\alpha\,R_\alpha\,,\quad L_\alpha^*\,J\,R_\beta \,=\, \delta_{\alpha\beta}\,,\quad \forall\alpha,\beta\in\{1,\ldots,N\}\,.$$

Then

$$J\,H\,L_\alpha \,=\, \overline{\omega}_\alpha\,L_\alpha\,,\quad \forall\alpha\in\{1,\ldots,N\}\,.$$

Proof. Let us denote by Ω the diagonal matrix of coefficients ω_α, by R the matrix of column vectors R_α, and by L^* the matrix of row vectors L_α^*. By assumption, we have

$$H\,R \,=\, -\,J^{-1}\,R\,\Omega\,,\quad L^* \,=\, R^{-1}\,J^{-1} \,=\, -\,R^{-1}\,J\,,$$

hence $L^*\,H\,R \,=\, \Omega$, or equivalently, writing $R \,=\, -\,J\,(L^*)^{-1} \,=\, J^*\,(L^*)^{-1}$,

$$L^*\,H\,J^*\,(L^*)^{-1} \,=\, \Omega\,.$$

Taking the adjoint we get $L^{-1}\,J\,H\,L \,=\, \Omega^*$, which is the matrix formulation of the claimed result. □

Lemma A.4. *Let* $\mathbf{H}$ *be a Hermitian* $(2p) \times (2p)$ *matrix, and*

$$\mathbf{J} \;=\; \begin{pmatrix} \mathbf{0}_p & I_p \\ -I_p & \mathbf{0}_p \end{pmatrix},$$

such that $\mathbf{JH}$ *is hyperbolic. Assume that* $(\mathbf{R}_1, \ldots, \mathbf{R}_p)$ *is a basis of* $\mathbb{E}_s(\mathbf{JH})$*, the stable subspace of* $\mathbf{JH}$*, and denote by* $(\mathbf{R}_{p+1}, \ldots, \mathbf{R}_{2p})$ *the basis of* $\mathbb{E}_u(\mathbf{JH})$*, the unstable subspace of* $\mathbf{JH}$ *characterized by*

$$\mathbf{R}^*_{p+\beta}\,\mathbf{J}\,\mathbf{R}_\alpha \;=\; \delta_{\alpha\beta}\,.$$

We decompose these vectors of $\mathbb{C}^{2p}$ *as*

$$\mathbf{R}_\alpha \;=\; \begin{pmatrix} R_\alpha \\ S_\alpha \end{pmatrix}, \quad R_\alpha \in \mathbb{C}^p\,,\; S_\alpha \in \mathbb{C}^p\,,$$

Then the following properties are equivalent

i). $\mathbb{E}_s(\mathbf{JH}) \cap (\mathbb{C}^p \times \{0_p\}) \neq \{0_{2p}\}$

ii). $\exists (\lambda_1, \ldots, \lambda_p) \in \mathbb{C}^p \backslash \{(0, \ldots, 0)\}$ *such that*

$$\sum_{\alpha=1}^{p} \lambda_\alpha\, S_\alpha = 0\,.$$

iii). $\exists L \in \mathbb{C}^p \backslash \{0\}$ *such that* $L^*\, S_\alpha = 0$ *for all* $\alpha \in \{1, \ldots, p\}$,

and the link between ii) and iii) is

$$\overline{\lambda}_\alpha \;=\; L^*\, S_{p+\alpha}\,, \quad \alpha \in \{1, \ldots, p\}\,.$$

Proof. The equivalence between i) and ii) is straightforward. Recall that $\mathbb{E}_s(\mathbf{JH})$ is of dimension p because the eigenvalues of $\mathbf{JH}$ form pairs $(\omega, -\overline{\omega})$. Furthermore, considering the adjoint systems of ODEs $U' \;=\; \mathbf{JH}\, U$ and $Z' \;=\; \mathbf{HJ}\, Z$, which are such that $(Z^*U)' = 0$ along their solutions, we easily see that the stable subspaces $\mathbb{E}_s(\mathbf{JH})$ and $\mathbb{E}_s(\mathbf{HJ})$ of respectively $\mathbf{JH}$ and $\mathbf{HJ}$ are orthogonal to each other,

$$\mathbb{E}_s(\mathbf{JH}) \;=\; \mathbb{E}_s(\mathbf{HJ})^\perp\,,$$

and if $(\mathbf{R}_1, \ldots, \mathbf{R}_p)$ is a basis of $\mathbb{E}_s(\mathbf{JH})$ then $(\mathbf{JR}_1, \ldots, \mathbf{JR}_p)$ is a basis of $\mathbb{E}_s(\mathbf{HJ})$. Therefore, i) is equivalent to the existence of $(\nu_1, \ldots, \nu_p) \in \mathbb{C}^p \backslash \{(0, \ldots, 0)\}$ such that

$$(\overline{\nu}_1, \ldots, \overline{\nu}_p, 0, \ldots, 0)\, \mathbf{JR}_\alpha \;=\; 0 \quad \forall \alpha \in \{1, \ldots, p\}\,,$$

which equivalently reads

$$(\overline{\nu}_1, \ldots, \overline{\nu}_p)\, S_\alpha \;=\; 0 \quad \forall \alpha \in \{1, \ldots, p\}\,,$$

and is nothing but iii) with $L = (\nu_1, \ldots, \nu_p)^T$. Then the link between ii) and iii) is that we can choose the λ's and ν's such that $(\nu_1, \ldots, \nu_p, 0, \ldots, 0)^T \;=\; -\sum_{\alpha=1}^{p} \lambda_\alpha\, \mathbf{R}_\alpha$, which yields

$$\lambda_\alpha \;=\; -\,\mathbf{R}^*_{p+\alpha}\, \mathbf{J} \begin{pmatrix} L \\ 0 \end{pmatrix} \;=\; S^*_{p+\alpha}\, L\,,$$

or equivalently, $\overline{\lambda}_\alpha \;=\; L^*\, S_{p+\alpha}$. ☐

Multilinear analysis.

Lemma A.5. *Assume that* $(r_1, \dots r_p)$ *is a family of vectors of* $\mathbb{C}^p$ *depending smoothly on a parameter* $\tau \in \mathbb{R}$*, and consider the mapping*

$$\Delta : \tau \mapsto \det(r_1(\tau), \dots, r_p(\tau)) .$$

Assume that Δ *vanishes at order one at some* $\underline{\tau}$*. Then the* $p \times p$ *matrix* $(r_1(\underline{\tau}), \dots, r_p(\underline{\tau}))$ *is of rank* $(p-1)$*.*

Proof. Up to a reordering of the vectors r_α, the fact that $\Delta(\underline{\tau}) = 0$ implies the existence of $(p-1)$ complex numbers λ_α such that

$$r_1(\underline{\tau}) \;=\; \sum_{\alpha=2}^{n} \lambda_\alpha \, r_\alpha(\underline{\tau}) .$$

Next we observe that for all τ,

$$\Delta(\tau) = \det(\widetilde{r}_1(\tau), r_2(\tau), \dots, r_p(\tau)) , \quad \widetilde{r}_1 \;:=\; r_1 \;-\; \sum_{\alpha=2}^{n} \lambda_\alpha \, r_\alpha .$$

This modification of the first vector simplifies the computation of the derivative of Δ (the same trick is widely used in Evans functions calculations), which readily reduces at $\underline{\tau}$ to

$$\Delta'(\underline{\tau}) \;=\; \det(\widetilde{r}_1(\underline{\tau}), r_2(\underline{\tau}), \dots, r_p(\underline{\tau})) .$$

For this derivative to be nonzero, the vectors $(r_2(\underline{\tau}), \dots, r_p(\underline{\tau})$ must be independent. $\square$

The converse is not true. Take for instance $p = 2$,

$$r_1(\tau) = \begin{pmatrix} \tau + 1 \\ 1 \end{pmatrix} , \; r_2(\tau) = \begin{pmatrix} -1 \\ \tau - 1 \end{pmatrix} ,$$

then $\Delta(\tau) = \tau^2$ vanishes at order 2 at zero, whereas $(r_1(0), r_2(0))$ is obviously of rank 1.

Acknowledgments Research of the second author was supported by the French Agence Nationale de la Recherche, contract ANR-08-JCJC-0132-01. The second author warmly thanks the Institut Camille Jordan in Lyon and the laboratoire Jean Leray in Nantes for their kind hospitality during the period in which this work was initiated.

References

[1] G. Alì and J. K. Hunter. Nonlinear surface waves on a tangential discontinuity in magnetohydrodynamics. *Quart. Appl. Math.*, 61(3):451–474, 2003.

[2] G. Alì, J. K. Hunter, and D. F. Parker. Hamiltonian equations for scale-invariant waves. *Stud. Appl. Math.*, 108(3):305–321, 2002.

[3] T. B. Benjamin. Impulse, flow force and variational principles. *IMA J. Appl. Math.*, 32(1-3):3–68, 1984.

[4] S. Benzoni-Gavage. Local well-posedness of nonlocal Burgers equations. *Differential Integral Equations*, 22(3-4):303–320, 2009.

[5] S. Benzoni-Gavage, J.-F. Coulombel, and S. Aubert. Boundary conditions for Euler equations. *AIAA Journal*, 41(1):56–63, 2003.

[6] S. Benzoni-Gavage and M. Rosini. Weakly nonlinear surface waves and subsonic phase boundaries. *Comput. Math. Appl.*, 57(3-4):1463–1484, 2009.

[7] S. Benzoni-Gavage, F. Rousset, D. Serre, and K. Zumbrun. Generic types and transitions in hyperbolic initial-boundary-value problems. *Proc. Roy. Soc. Edinburgh Sect. A*, 132(5):1073–1104, 2002.

[8] S. Benzoni-Gavage, D. Serre. *Multidimensional hyperbolic partial differential equations*. Oxford Mathematical Monographs. Oxford University Press, 2007.

[9] Sylvie Benzoni Gavage, Jean-Francois Coulombel, and Nikolay Tzvetkov. Ill-posedness of nonlocal Burgers equations. *Advances in Mathematics*, To appear. Preprint at `http://hal.archives-ouvertes.fr/hal-00491136/en/`.

[10] J.-F. Coulombel and O. Guès. Geometric optics expansions with amplification for hyperbolic boundary value problems: linear problems. *Ann. Inst. Fourier (Grenoble)*, 60(6):2183–2233, 2010.

[11] R. Hersh. Mixed problems in several variables. *J. Math. Mech.*, 12:317–334, 1963.

[12] J. K. Hunter. Nonlinear surface waves. In *Current progress in hyberbolic systems: Riemann problems and computations (Brunswick, ME, 1988)*, volume 100 of *Contemp. Math.*, pages 185–202. Amer. Math. Soc., 1989.

[13] J. K. Hunter. Short-time existence for scale-invariant Hamiltonian waves. *J. Hyperbolic Differ. Equ.*, 3(2):247–267, 2006.

[14] H.-O. Kreiss. Initial boundary value problems for hyperbolic systems. *Comm. Pure Appl. Math.*, 23:277–298, 1970.

[15] A. Marcou. Rigorous weakly nonlinear geometric optics for surface waves. *Asymptotic Anal.*, 69(3-4):125–174, 2010.

[16] G. Métivier. The block structure condition for symmetric hyperbolic systems. *Bull. London Math. Soc.*, 32(6):689–702, 2000.

[17] P. J. Olver. Hamiltonian and non-Hamiltonian models for water waves. In *Trends and applications of pure mathematics to mechanics (Palaiseau, 1983)*, volume 195 of *Lecture Notes in Phys.*, pages 273–290. Springer, 1984.

[18] Peter J. Olver. *Applications of Lie groups to differential equations*, volume 107 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

[19] D. F. Parker. Waveform evolution for nonlinear surface acoustic waves. *Int. J. Engng Sci.*, 26(1):59–75, 1988.

[20] D. F. Parker and F. M. Talbot. Analysis and computation for nonlinear elastic surface waves of permanent form. *J. Elasticity*, 15(4):389–426, 1985.

[21] R. L. Pego and M. I. Weinstein. Eigenvalues, and instabilities of solitary waves. *Philos. Trans. Roy. Soc. London Ser. A*, 340(1656):47–94, 1992.

[22] M. Sablé-Tougeron. Existence pour un problème de l'élastodynamique Neumann non linéaire en dimension 2. *Arch. Rational Mech. Anal.*, 101(3):261–292, 1988.

[23] D. Serre. Second order initial boundary-value problems of variational type. *J. Funct. Anal.*, 236(2):409–446, 2006.